Universality and critical phenomena in string defect statistics

Karl Strobl and Mark Hindmarsh

Centre for Theoretical Physics, University of Sussex, Brighton BN1 9QH, United Kingdom

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The idea of biased symmetries to avoid or alleviate cosmological problems caused by the appearance of some topological defects is familiar in the context of domain walls, where the defect statistics lend themselves naturally to a percolation theory description, and for cosmic strings, where the proportion of infinite strings can be varied or disappear entirely depending on the bias in the symmetry. In this paper we measure the initial configurational statistics of a network of string defects after a symmetry-breaking phase transition with initial bias in the symmetry of the ground state. Using an improved algorithm, which is useful for a more general class of self-interacting walks on an infinite lattice, we extend the work [in M. Hindmarsh and K. Strobl, Nucl. Phys. B **437**, 471 (1995)] to better statistics and a different ground-state manifold, namely, $\mathbb{R}P^2$, and explore various different discretizations. Within the statistical errors, the critical exponents of the Hagedorn transition are found to be quite possibly universal and identical to the critical exponents of three-dimensional bond or site percolation. This improves our understanding of the percolation theory description of defect statistics after a biased phase transition. We also find strong evidence that the existence of infinite strings in the Vachaspati-Vilenkin algorithm [T. Vachaspati and A. Vilenkin, Phys. Rev. D **30**, 2036 (1984)] is generic to all (string-bearing) vacuum manifolds, all discretizations thereof, and all regular three-dimensional lattices. [S1063-651X(97)02401-X]

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INTRODUCTION

Many symmetries in nature are not exact, including internal symmetries in field theories. A simple example of an approximate symmetry is a spin system in an external field, e.g., a nematic liquid crystal with diamagnetic molecules [1]. In particle physics, an example of an approximate symmetry is the Peccei-Quinn symmetry $U(1)_{PQ}$, associated with the axionic degree of freedom, which has the degeneracy of its ground-state manifold lifted at the QCD scale.

Condensed-matter systems, as well as the vacuum in the early Universe, are, in the process of being cooled, subject to phase transitions. If these transitions are accompanied by symmetry breaking, they may lead to the formation of defects: domain walls, strings or vortices, monopoles, or a combination of these, depending on the topology of the set of equilibrium states after the phase transition. In cosmology such defects are associated with internal symmetries of a field theory, while in condensed-matter systems there are defects associated with the rotational symmetry of the ground state. Such defects in a nematic liquid crystal are called disclinations [1]. Along a closed path around line disclinations, the orientation of the molecules rotates by an angle π, while in a point defect the molecules are in a radial point defect or ''hedgehog'' configuration (or a continuous deformation thereof), directed away from a central point. On the central points of the disclinations the molecules cannot have any alignment directions.

Topological defects form through what in cosmology is called the Kibble mechanism [2]: The simple requirement of causality prevents regions of the Universe that are separated by more than twice the horizon distance $\int_0^t a(t')dt'$ from being correlated. The actual correlation length can of course be much smaller. It is this lack of correlation that allows the initial conditions to trap topological defects after phase tran-

sitions. Defects are also a convenient and compelling way to seed large-scale structure formation in the early Universe. A crucial ingredient for the usual string-seeded structure formation scenario is that the string configurations develop a scaling solution [3,4], which in turn seems to depend on the initial scale invariance of the network. The presence or absence of infinite strings (or, in a closed universe, of strings that wrap around it) also seems to affect the string density in the scaling solution [5]. Moreover, the question whether any respectable fraction of the string mass ought to be in infinite strings is still controversial and far from being decidable by analytical means: The usual Vachaspati-Vilenkin algorithm yields lattice-dependent results for this fraction, which can be attributed to the assumption of an unphysical lattice-dependent lower cutoff in the loop length distribution [6,7]. It has been claimed that a complete absence of infinite strings can be achieved by similar algorithms on generalized graphs, corresponding to an irregular discretization of allowed string positions, obtained through modeling the collisions of true vacuum-phase bubbles after a first-order phase transition [8]. However, the reason for the very small fraction of infinite strings in [8] has not yet been identified, and various proposals are on the market [9,10].

What is desired for strings is a cosmological disaster for domain walls: Infinite walls would come to dominate the expansion very quickly, which is incompatible with stronger bounds from the cosmic microwave background [11]. A convenient way to escape this problem has been to make the symmetry between the disjoint (sets of) vacuum states an approximate one [12,13]. In particle physics such adjustments are possible and in fact necessary, for instance, within attempts to explain the family hierarchy [14]. Similar biases in other symmetries can affect the configurational statistics for all kinds of topological defects. Attempts to circumvent the monopole problem by allowing monopoles to annihilate with antimonopoles at a sufficient rate (facilitated through an acclaimed tendency for the two to accompany each other)

have been made [15]; however, effects that soft symmetry breaking might have to facilitate the monopole antimonopole correlations to a higher degree have not been considered. In this paper we will not address monopoles, but attempt to establish such a description in the case of cosmic strings and topological line defects in solids, i.e., for softly spoiled symmetries with a nontrivial first homotopy group. The technical details of the lattice algorithm and the proof that it conserves (in most cases) the essential invariants of the continuum theory are presented in a separate paper [16].

In [6] it was pointed out that the existence of infinite strings may be understood as a percolation phenomenon and the associated critical behavior can be observed at a transition into a phase where the string network exhibits only loops. In this paper we show that this is indeed a percolation transition and can only be obtained, once the vacuum manifold and the discretization thereof and of space-time are chosen, by biasing the symmetry of the ground state. Although still no proof is available, we find strong evidence that the existence of infinite strings in the Vachaspati-Vilenkin algorithm at perfect vacuum symmetries is generic to all vacuum manifolds, all discretizations thereof, and all regular three-dimensional lattices.

We also find that the critical exponents for configurational parameters near the percolation threshold are universal for different vacuum manifolds and identical (to within our statistical errors) to the corresponding critical exponents in standard bond or site percolation theory. For the case of an $\mathbb{R}P^\infty$ symmetry and a minimally discretized U(1) symmetry, plausibility arguments for this correspondence are brought forward.

Section I introduces the well-known scaling concept and methods to make it break down through a bias in the vacuum symmetry. There, and in Sec. II, we point out some aspects of the scaling concept, which, to our knowledge, have not been mentioned in the existing literature on the subject. In particular, a clear distinction between the different manifestations of scaling in the loop and the infinite-string ensemble is made and a correlation to scaling in percolation theory is illustrated. In Sec. II we show ways to control and estimate statistical errors and present results of measurements for perfect vacuum symmetries. Section III explains the theoretical basis on which one expects the percolation transition to occur. Section IV presents results for this Hagedorn-like transition at which the infinite strings start to appear (as one decreases the bias in the symmetry). We extract critical exponents associated, e.g., with the divergence of the average loop length, a suitably defined susceptibility, and a correlation length (for correlations in the string configurations, not the vacuum field). Compared to Ref. [6], the accuracy of the results is greatly improved, results for the $\mathbb{R}P^2$ symmetry are different, and statistical errors are estimated. Section V discusses issues of the universality of the critical exponents and a percolation theory understanding of the Hagedorn transition is developed. A renormalization-group (RG) understanding of the scaling concept is developed and problems with the RG method in calculating the critical exponents are addressed, as are cosmological implications.

I. FUNDAMENTALS OF STRING STATISTICS

A. Numerical methods

We will evaluate numerically the statistics of cosmic U(1) strings (or, equivalently, vortices of superfluid ^{4}He) and of $\mathbb{R}P^2$ strings (e.g., line disclinations in nematic liquid crystals), in lattice-based simulations of the Kibble mechanism. The numerical method is presented in Ref. [16] and contains perhaps essential improvements to the usual Vachaspati-Vilenkin (VV) algorithm [17]. The most important improvement in our calculations is that our lattice size is formally infinite, i.e., we can avoid specifying any boundary conditions, and can trace much longer strings with a given amount of computer memory than was possible before. The VV algorithm in general discretizes space such that the lattice spacing corresponds to the smallest physical length beyond which field values can be considered to be uncorrelated, i.e., the lattice spacing a is of the order of, but perhaps slightly larger than, the correlation length ξ of the field at the symmetry-breaking phase transition, but certainly no larger than the cosmological horizon. Details of the field dynamics are then inessential to the statistical properties of a large ensemble of such lattices and vacuum field values are assigned randomly and independently at each space point to create a Monte Carlo ensemble of field configurations on the lattice. It should be pointed out that the regular lattice we use has been criticized, as it does not allow variations in the size of correlated domains. In particular, some of the results of Borrill's simulations [8] are rather different from ours, for reasons that are still poorly understood. However, they may well suffer from important finite-size effects.

Line defects are then considered to have formed if a closed walk along lattice links maps, through the field map, onto a noncontractible loop on the vacuum manifold. The assumption of a "geodesic rule" [18,19] for the interpolation of the field between the lattice points is not only intuitively acceptable, but in this formalism it is also essential in order to guarantee string flux conservation, an essential symmetry of the problem [16,7]. References [16,7] also prove that only the dual lattice to the tetrakaidekahedral lattice can preserve a uniqueness in the identification of the paths of single strings and rotational symmetry of the Monte Carlo ensemble at the same time. This lattice has been used in this context in Refs. [6,7,20] and to study simulations of monopoles and textures [15,21].

B. Long-range correlations in topological defects

Strings are usually modeled by random walks, either Brownian or self-avoiding. As usual in the literature, we will use the abbreviation SAW to mean self-avoiding *random* walk (there are obviously infinitely many ways of building self-avoiding walks, each leading to possibly quite different statistics; straight walks, for example, are self-avoiding but obviously exhibit quite different statistics to SAWs in dimensions higher than one). A self-avoiding random walk models an excluded-volume effect and is known to apply well to polymers [1]. However, it is not clear that either kind of walk represents the configurational statistics of cosmic strings or superfluid vortices, for there are long-range interactions that could change the Hausdorff dimension. That there are superhorizon correlations in the configurations of topological defects is not exactly new [15]. In the case of cosmic strings it can be demonstrated by arriving at a simple contradiction when assuming no long-range correlations. Take a closed circular walk through three-space spanning

many correlation volumes. How many strings does one expect to encircle with such a walk? Since there is a well-defined string density per correlation area, the number of encircled strings should increase proportionally with the area A enclosed by the walk. If they are uncorrelated, the net flux through this area will be partially canceled by strings of opposite orientation and one expects an average net flux of around $\sqrt{A}$ going through the loop formed by our walk. On the other hand, the net flux is given also by seeing how often the field winds around U(1) while one follows the walk. This number, however, is expected to increase as the square root of the length of the walk, since the field values are uncorrelated on some length scale that is small compared to the size of the loop and many of the windings will cancel out. One therefore has to conclude that, if the Kibble mechanism is responsible for the formation of strings (i.e., if the *field values* are uncorrelated beyond some scale initially), the string network will have long-range correlation, favoring, for instance, flux cancellation for oriented strings on large surfaces. One should therefore expect deviations from Brownian behavior in the string statistics. This is also favored, because a cosmic string clearly *is* self-avoiding: Because a string is defined by the topology of the field map, it will always follow the same way again, once it has turned back onto itself. A cosmic string is therefore always forming a loop or has to be infinite and self-avoiding. This does not mean, however, that the string's configurational statistics are those of a self-avoiding *random* walk, because the nature of the self-avoidance is dictated by the field map. A SAW shows correlations only on very short scales (typically scales of the lattice constant). One of the reasons why strings also cannot be randomly self-avoiding is that the field map carries the memory of the position of all the other strings, which cannot be crossed. We will show that neither a Brownian walk nor a random self-avoiding random walk models cosmic strings accurately and that the walk statistics depend on the vacuum manifold creating the strings.

C. Scaling hypothesis

A U(1) string in the Vachaspati-Vilenkin algorithm on a tetrahedral lattice is self-avoiding [7], irrespective of the discretization (if any) of U(1) used in the algorithm. One might therefore expect the network of cosmic strings to have the statistical properties of a self-avoiding random walk. A SAW builds up an excluded volume as it follows its path, which is, in a statistical sense, spherically symmetric and clustered around the origin [22]. The SAW therefore has a stronger tendency to move away from the origin than the Brownian walk, which is allowed to intersect itself arbitrarily often. This property is expressed in the fractal dimension D of the walk, which is the exponent relating the average string length l between two points on the same string to their relative distance R by

$$l \propto \langle R \rangle^D, \qquad (1)$$

where the angular brackets denote some averaging procedure over a large ensemble of walks. We show in another paper that several different averaging procedures, in particular the ones of the kind

$$\langle R \rangle = \lim_{N \to \infty} \left(N^{-1} \sum_{i=1}^{N} |R^n| \right)^{1/n} = (\langle |R^n| \rangle)^{1/n},$$

produce the same results for the fractal dimension, such that in particular $l \propto \langle |R|^D \rangle \propto \langle |R| \rangle^D$ [23]. It is well known that the dimension for a Brownian walk is $D = 2$, and for a self-avoiding random walk in three dimensions it is $D = 1/\nu = 1/(0.5877 \pm 0.0006)$ (see Ref. [24] and references therein for a summary of different methods used to obtain that result). However, the original string formation simulations [17,25] are consistent with $D = 2$. The reason for this was seen in the fact that they simulated a dense string network: A single string, as we trace out its path, experiences a repulsion from all of the segments of other strings, which do not have any statistical bias towards the origin. Therefore the repulsion from the forbidden volume will also have no directional bias. Thus the fractal dimension of the string could also be argued to be (close to) 2, although the string is self-avoiding. In polymer physics, this effect has been known for some time to occur in a dense solution of polymers [1]. In a statistical sense, the network of cosmic strings was argued to be equivalent to a dense network of polymers [26]. A polymer in a dilute solution will exhibit the configurational statistics of a self-avoiding random walk, while in a dense solution of polymers, each one has the structure of a Brownian random walk. Thus, taking this lesson from polymer physics, one would expect the scaling of the string size R with length l in the initial configuration of cosmic strings to correspond to a SAW on scales smaller than the mean separation between different strings and to a Brownian walk on scales larger than this. In the cosmic string case, however, the mean separation is of the order of the correlation length itself, which is the same as the lattice spacing. So we expect the bias towards a SAW to fall off with distance roughly as fast as the lattice discretization errors, which makes this short-distance effect immeasurable. We shall anticipate the results of the following sections: the fractal dimension of a string at formation, in general, is not the same as for a Brownian walk. It is only for U(1) strings that measurements are consistent with the exact value of 2, in the extremely long-distance limit ($\approx 10^5 - 10^6$ lattice units). Of the other strings that have been measured, none have fractal dimensions higher than the U(1) strings, but all have distinctly larger D than the SAW.

As is customary, we can introduce the scaling hypothesis in order to estimate a few other properties of the string network. It states that, in terms of its statistical properties, the string network looks the same on all scales much larger than the correlation length of the vacuum field. Scale invariance is phenomenologically the same as the existence of a large-scale (IR) renormalization-group fixed point. However, renormalization-group arguments for topological objects are hard to find. To our knowledge, there exists no analytic work that would lend firm support to the scaling hypothesis. In fact, most analytic work gets intractable if the scaling hypothesis is not put in *a priori*. One would expect a proof of the scaling hypothesis to contain RG arguments. We will develop percolation theoretical RG arguments in favor of this hypothesis in Appendix B. With the scaling hypothesis, the expected distribution of closed loops can be derived [27].

From dimensional arguments, the number of closed loops with size from R to $R+dR$ per unit volume can be written as

$$dn = f\left(\frac{R}{\xi}\right)\frac{dR}{R^4}. \qquad (2)$$

If the system is scale invariant, the distribution should be independent of the correlation length ξ and one expects

$$dn \propto R^{-4}dR. \qquad (3)$$

The length distribution of loops for strings with a fractal dimension of D is therefore

$$dn \propto l^{-b}dl, \qquad (4)$$

with

$$b = 1 + 3/D \qquad (5)$$

or, more generally, $1+d/D$, with d being the dimension of the space that the walk is embedded in. It was originally expected [17] that it follows from scale invariance that there should be no infinite strings. This turned out not to be the case, since, as we will discuss in Sec. IIB, ensembles of infinite strings and ensembles of loops manifest scale invariance in entirely different ways, namely, in the validity of the Eqs. (1) and (5), respectively. Infinite strings can still look statistically the same on all scales much larger than the lattice spacing: A Brownian walk is scale invariant and has a nonzero probability not to return to the origin in $d>2$ dimensions. The origin of the scale invariance of the string network seems to be connected with the absence of long-range correlations in the order parameter [17]. However, scale invariance does not necessarily imply that the network is Brownian as originally stated. Strictly speaking, scale invariance holds when all the scaling properties of a network, such as Eqs. (1) and (4), are power laws: Only power laws do not change upon linear rescaling of the variables. In this sense, scaling is satisfied whenever Eqs. (1) and (4) hold. However, to make scaling also work in spite of finite-size effects prohibiting us from identifying the very long loops, Eq. (5) is taken as the manifestation of scale invariance. This is plausible: Equation (5) implies that loops exhibit (on scales much larger than the lattice spacing but much smaller than the loop size) the same fractal dimension as infinite strings, so that on scales where one counts some number of loops wrongly as infinite strings, the distinction between the two becomes unnecessary. In this sense, Eq. (5) is a more stringent definition of scale invariance because it allows one to be ignorant about effects on scales that a particular observation may not reach. If we defined scale invariance by some omniscient observer that can distinguish loops even if they exceed the observed scale in size, then there is no reason for the exponents of the loop distribution to be in any relation to the exponents of the distribution in infinite strings. One, of course, does not need $D=2$ in order to have a scale-free distribution of loop sizes R. It is important to note that, because of Eqs. (1) and (2), although they are the criteria for scale invariance, there are some observables that are not scale invariant if they happen to be dependent on the UV cutoff. An example of this is the fraction of string mass in
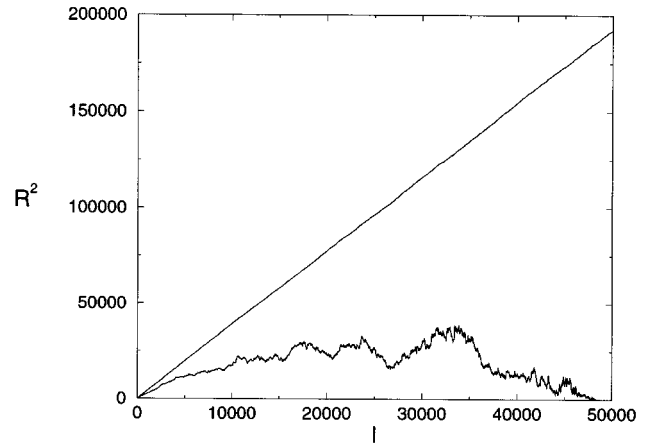

FIG. 1. Relation between the average distance of a string element from the origin and the string length walked until arriving at this element. The upper line represents averages over the "infinite" strings only (i.e., strings that survive up to the length $\Lambda = 50\,000$ where this particular measurement was stopped). The lower line represents averages over the string loops and has high statistical errors on the long-loop end, because of the low number of loops, and even higher systematic errors, because the ratio of loops wrongly counted as infinite to the correctly counted loops increases. The averages were taken over 10 000 strings, 6334 of which happened to be infinite. Only 54 loops survived up to length 10 000 and only 10 to length 30 000. One sees that the relation for R^2 vs l for the infinite string part is almost perfectly linear, suggesting Brownian statistics. The vacuum manifold U(1) was discretized by three equidistant angles.

loops, as discussed in [6] and item (iii) in Sec. II A. Whatever numbers one gets for these quantities are probably unphysical since there is no known algorithm (least of all VV-type algorithms) that would tell us what the physical UV cutoff on the loop size distribution Eq. (4) should look like. Scale invariance can hold only in the limit $l \gg \xi$.

If $D=2$, one would expect a linear relationship between walk length and average R^2, which would then, if the probability distribution for ending up at a point $\vec{x}$ after l steps is Gaussian, be interpretable as the average σ^2 of the distribution. All this is familiar from the Brownian walk, and measurements seem to indicate that, in the case of a U(1) symmetry, we are close to such statistics. Figure 1 presents the linear-linear graph of R^2 vs the walk length l, which can be seen to be an almost perfectly linear relation. The measurements in Figs. 1 and 2 are made using a discretization of U(1) by three equidistant angles. [In a sloppy way, we could say we discretize U(1) by $\mathbb{Z}_3$. This only is correct as far as the allowed vacuum angles are concerned, but may be misleading since in an actual $\mathbb{Z}_3$ symmetry the lines that we identify as geodesics on U(1) would be associated with a finite vacuum energy, i.e. they would be crossing domain walls. It is more correct to say that we triangulate the vacuum manifold as well as space: in this case with three vertices and three edges, joining adjacent points. This automatically encodes the geodesic rule. This distinction seems trivial, but it allows an easier generalization to, e.g., discretizations of $\mathbb{R}P^N$).] Such measurements were made in [6] and we complete results from there. In particular, we present a
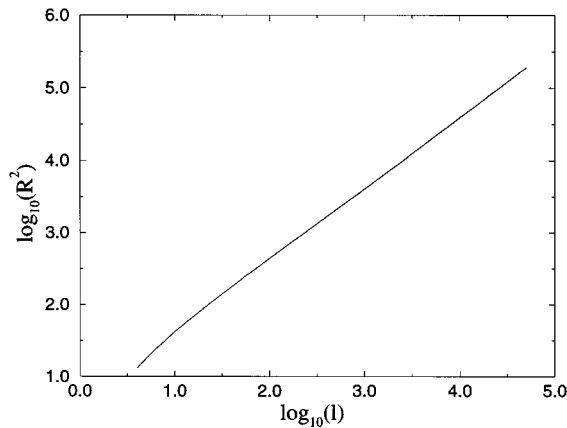
FIG. 2. log-log plot of the ''infinite string'' contribution in Fig. 1. We see that the short-length limit has lattice discretization errors and that a true power law is approached only for $l \gtrsim 30$. The measurements are listed in Table II.

much better error analysis here. Results are represented split between the infinite-string part and the loop contribution. This is a procedure we will follow throughout this paper, in analogy to conventions in percolation theory, and we will show that it is in fact necessary to do so.

II. RESULTS FOR PERFECT SYMMETRIES

A. Elimination and estimation of errors

Before we turn to the results, a few words of caution are in order. Among those, we include an explanation of how we arrive at error estimates for the statistical error.

(i) Because of the nature of the Monte Carlo averaging, there are two big sources of error for very long loops (i.e., the longest ones permitted in the simulations). We follow every string until it hits a certain upper limit of the string length $l \leqslant \Lambda$ or until it returns to the origin, whatever happens first. If it does not return to the origin until we have reached the length Λ, it is counted towards the ''infinite-string'' ensemble. This does not introduce too many problems for the averaging over infinite strings (as long as there are many) because of the nature of Eq. (3), which ensures that only very few of the strings surviving up to length Λ are actually wrongly counted as infinite. For the loop distribution, which has only very few strings in this regime, the statistical errors are huge (in the end, usually just before we reach $l = \Lambda$, we even ''average'' over one string only), but the systematic errors in this regime are equally bad because there is a finite number of strings that *should* be in the loop distribution but are not identified as loops. This drives the measured R^2 to zero at the length where the longest of the correctly identified loops closes (compare the lower line in Fig. 1). Extraction of configurational exponents on the loop distribution will therefore be defined through fitting appropriate curves that approach the actual measurements asymptotically in the intermediate-length regime only.

(ii) A word of caution is also necessary for the short-length limits. As seen, for example, in Fig. 2, scaling is not satisfied in the short-walk-length regime. This is due to two sources of error. First, there are obviously lattice discretization artifacts. Second, at small distances the excluded-

volume effect (from the self-avoiding nature of the string) is still turned on, with a directional bias away from the origin. Eventually, repulsion from all other pieces of string should even out with repulsion from pieces of the same string, with no directional bias at all, but this happens only at some distance from the origin. We will allow for this source of error by cutting off the low-length regime at lengths between 10 and 1000 lattice units whenever we measure configurational exponents from the Monte Carlo ensembles. A cutoff of 20 is usually sufficient for extracting the exponents c and b in Eq. (8), while 500 is very good, and still perfectly practical, for extracting the fractal dimension D on the infinite string part. Which cutoff to choose is decided on a case by case basis by observing where a cutoff independent measurement can be obtained.

(iii) Frequently, we will not quote the fraction of the total string mass found in loops. Such numbers are meaningless unless there is a lower cutoff for loop lengths much larger that one in units of correlation lengths (in which case one gets very little mass in loops anyway). This parameter is lattice dependent [26], even for simple Brownian random walks or self-avoiding random walks. This is partly due to different coordination numbers of different lattices. (For example, every vertex on the tetrakaidekahedral lattice is connected to four lattice links, while this number is six in the simple cubic case. There are more possibilities for the string to ''stray'' in the simple cubic case.) This is connected with another lattice dependence of this number: The length of the lattice links in the tetrakaidekahedral case is $a\sqrt{2}/4$, with a the edge length of the underlying bcc lattice, while the correlation length is between $a\sqrt{3}/2$ and a (since every link borders three tetrakaidekahedra, we need to take all the distances between those three as representatives of a correlation length; two pairs have the distance $a\sqrt{3}/2$, while one pair is separated by a). Henceforth, when we refer to ''walk lengths in lattice units'' we mean in units of $a\sqrt{2}/4$, which is the edge length of the tetrakaidekahedral lattice. Since the smallest allowed loops in both lattices consist of four links, the tetrakaidekahedral lattice allows much smaller loops (in units of correlation lengths) than the simple cubic lattice. According to Eq. (3), we expect a large contribution to the total string mass to be in very small loops, so that on a tetrakaidekahedral lattice the total string mass in loops will be considerably higher than on the cubic lattice. The problem of the lattice dependence of the mass fraction in loops also reflects a lack of knowledge about the physics involved in the production of small loops. Physically, one would expect a smooth cutoff for short loops, so that the very small loop contribution in Eq. (3) gets gradually suppressed. We do not know the form of this cutoff and we expect it to depend not only on dynamical details of the Kibble mechanism but also on thermal production mechanisms for string loops (which should be relevant right at the phase transition temperature, but quickly become subdominant as the Universe cools further). In any case, the physical relevance of knowing the exact contribution to the total string mass in small loops produced at a phase transition is highly questionable, as they disappear quickly in any case. This does not affect the physical relevance of the other data we can extract from Vachaspati-Vilenkin-type measurements because the long loops and infinite strings are not transient.

TABLE I. Statistical variances in measurements of the fractal dimension for ensembles of fewer and fewer strings. The statistical error for a large ensemble is the extrapolation of these values to the appropriate number of strings.

Number of strings	Number of ensembles	Average D	Standard deviation
1000	10	2.027	0.038
500	10	2.022	0.045
250	10	2.035	0.051
100	10	2.021	0.094

(iv) Finally, we need to explain how we arrived at the values for the statistical error. To estimate, for instance, the statistical error percentage of the fractal dimension D, measured for a total number of $\mathcal{N}_{measured}$ strings up to a length Λ, we take, for example, ten sets of $\mathcal{N}=\mathcal{N}_{measured}/10$ strings and measure the variance of the result, then take ten sets of $\mathcal{N}=\mathcal{N}_{measured}/20$ and ten sets of $\mathcal{N}=\mathcal{N}_{measured}/40$ strings, and so on. We then measure the variance of the results for all those sets and, under the assumption that the error behaves like a power of the size of the string ensemble, we extrapolate to an ensemble of $\mathcal{N}_{measured}$ strings. If all our measurables were Gaussian random variables for any sample within the ensemble, this power law would just be $\sigma^{-1} \propto \sqrt{\mathcal{N}}$, which motivates this approach. Since configurational exponents are normally not distributed in a Gaussian distribution within samples of the ensemble, we decided to allow a generalized power law for the variance. We present this method by the example of a U(1) manifold discretized by $N=3$ equidistant vacuum angles. Figure 2, the log-log plot of $\langle R^2 \rangle$ vs l, has a linear fit, suggesting

$$l = 0.232 \ R^{2.0212} \quad (N=3).$$

This was measured for $\mathcal{N}_{measured}=10\,000$ strings being allowed to reach the length $50\,000$ in lattice units (taking only the ''infinite'' strings and using a lower cutoff of 500 lattice units). Similar measurements on several ensembles with fewer strings yield the values in Table I.

On a log-log plot, the statistical variances may be fit by the expression

$$\sigma \approx 0.0146 \ (\mathcal{N}/\mathcal{N}_{measured})^{-0.383},$$

so that the σ expected in our measurement can be taken to be ≈ 0.015, which is simply the intercept of the linear fit in the log-log plot of the variance against $\mathcal{N}/\mathcal{N}_{measured}$, as displayed in Fig. 3.

B. Results for a perfect U(1) symmetry

We can now proceed to the presentation of the results. For a perfect U(1) symmetry, we have used a series of different discretizations of U(1), each consisting of $N=2^n-1$ equidistant angles. The range of such discretizations is from $N=3$, the lowest possible number of points on U(1) to give noncontractible contours, to $N=255$, a rather good approximation to continuous symmetry, as we shall see from the asymptotic behavior of the measurables for large N.

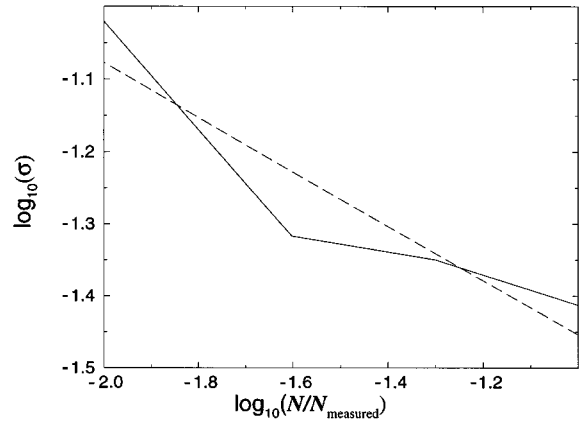FIG. 3. Variances in Table I. Extrapolating of the linear fit to $\mathcal{N}_{measured}$ yields an estimate for the statistical error in the original measurement, in this case $\sigma_{measured}=10^{-1.831} \approx 0.015$.

1. Loops have no fractal dimension

The linear fit to Fig. 4, the log-log plot of $\langle R^2 \rangle$ vs l, averaged over loops only, obviously requires some sensible upper cutoff much lower than Λ, and to some extent any measurement of the fractal dimension of the loop ensemble is cutoff dependent. Nevertheless, a fractal dimension of $D=2$ is inconsistent with any part of Fig. 4, as is any fractal dimension measured for the infinite-string contribution as listed in Table II. A typical subset of this loop ensemble, reduced to loops with $l \in [10,1000]$, gives $D=2.14$, whereas a typical infinite-string dimension for, say, strings with $\Lambda=50\,000$ is $D=2.02$. The reason for both this discrepancy and the cutoff dependence of the loop dimension is simple to understand. A single loop cannot be a fractal, but a single infinite string can be. As it turns out, the measurements allow a slightly stronger statement: The *average* loop is not a fractal, whereas the *average* infinite string is. The reason for this is simply the finite size of loops: A single infinite string approaches a scaling behavior asymptotically for large scales with no upper length scale arising, whereas a loop has a
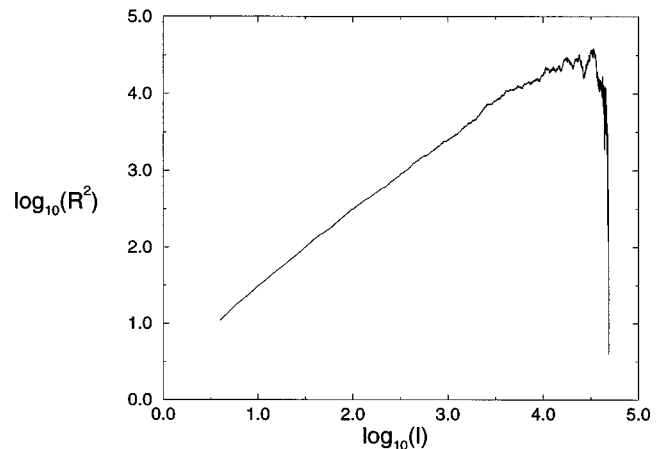

FIG. 4. log-log plot of the loop contribution in Fig. 1. Only the short-loop regime can be deemed useful, because there are very few long loops, making bad statistics. In addition to that, the counting is biased by the exclusion of all loops longer than Λ or shorter than any given length where the plot is read.

TABLE II. Fractal dimensions of an average string with different, increasingly finer, discretizations of U(1). The averages are over infinite strings only. The lower cutoff is $l=500$ in all ensembles.

Number N of discretization points on $\mathcal{M}$	Fractal dimension D	Number of strings	String length cutoff Λ
3	2.021 ± 0.015	10 000	50 000
7	2.022 ± 0.015	10 000	50 000
15	2.022 ± 0.015	10 000	50 000
31	2.022 ± 0.015	10 000	50 000
63	2.025 ± 0.015	10 000	50 000
127	2.013 ± 0.015	10 000	50 000
255	2.007 ± 0.025	3 000	1 000 000
255	2.055 ± 0.006	100 000	2 000

cutoff through its finite size. A finite object has a well-defined fractal dimension only on scales much smaller than the size of the object and much larger than the lattice spacing. This means that only very large loops will exhibit "scaling" and then only on a finite range of scales, which makes the scaling concept rather risky to depend on as far as the fractal dimension of the loops is concerned. We will therefore average over the infinite-string component of the string ensemble only, whenever it comes to extracting a fractal dimension. Figure 4 also suffers from the problem that the averages are only taken over those loops that actually survive up the given length. An "average loop" would simply not be there anymore after we have walked the length, say, $l=10$.

This does not invalidate Eqs. (3)–(6), as these specifically imply and require the finiteness of loops. We can sum this up in the following way: The scaling concept enters the loop distribution through the distribution of loop sizes rather than the average properties of a single loop, whereas the scaling properties of the infinite strings can be interpreted as properties of the average infinite string.

There is another qualifying statement to be made: If we remind ourselves how Eq. (4) was derived from Eq. (3), we have used the fractal dimension (in the cases to follow, the fractal dimension of the infinite-string ensemble) to derive properties of the loop ensemble. This is justified only because of a lower loop-size cutoff (in addition to the upper one, which ensures that the averaging is not over too small an ensemble) employed in measuring b. By having an appropriate lower cutoff, we make sure that we use this fractal dimension only for loops long enough to exhibit an intermediate-length scale on which fractal behavior can be approached, allowing us to use Eq. (1) in deriving (4) from (3). The measurements do then indeed seem to indicate that, with rather minor deviations, the intermediate-length regime of a properly chosen long-loop ensemble looks like an infinite-string ensemble with an intermediate-range upper cutoff and the scaling relation Eq. (4) holds.

2. Infinite-string ensemble

Using linear fits to log-log plots of R^2 vs l, with the error elimination and error estimation methods presented above, we arrive at measurements for the fractal dimension of

strings as presented in Table II. We see that the statistical errors are still larger than the discretization errors coming from a particular discretization of U(1) (although the string density, for instance, does depend on which discretization one uses). Table II also suggests systematic errors: The higher the upper cutoff Λ, the lower the measured fractal dimension. This effect has been observed, in a diffferent context, for U(1) strings with $N=3$ [28]. Except for the short strings with $\Lambda=2000$, the measurements, however, are consistent with each other. To decide whether the last line in Table II is actually the manifestation of systematic errors or not, let us investigate the possible sources for such a discrepancy. Either an excluded-volume effect is active for intermediate distances, making the strings slightly self-seeking, which forces us to accept a scale-dependent fractal dimension, which was one of the conclusions drawn for $N=3$, U(1) strings in Ref. [28]; we are counting too many strings that are to form loops eventually (but are not identified as loops, because of the cutoff Λ), which would naturally bias the fractal dimension towards a higher number for smaller Λ; or our averaging procedure introduces systematic errors. To explore which of these interpretations is the right one, we need to know how many of the strings that reach the length Λ are to be expected to close onto themselves again to form loops, i.e., we need to know the prefactor and the exponent involved in Eq. (4). Once we know the number of wrongly counted strings, we need to subtract their expected configurational parameters from the ensemble of infinite strings. As a first approximation, we assume that these strings have the average loop properties extracted from the loop ensemble. This will actually overcompensate for the effect that those strings lower the effective fractal dimension because they should, on average, of course have larger R^2 than the average strings that are actually counted as loops. This is easy to see in the large-l limit, where they are obviously not at $R^2=0$, whereas the strings that are counted as loops are arriving there for $l \rightarrow \Lambda$. Thus we will overestimate errors coming from this source. The procedure is now obvious: If the number density of *traced* loops is

$$dn = q l^{-b+1} dl, \qquad (6)$$

then the number of strings expected to exceed the length Λ, i.e., the number of uncounted loops, is

$$n_{\text{loop}}^{\text{unc}} = \int_{\Lambda}^{\infty} dn(l) = q \frac{\Lambda^{-b+2}}{b-2}. \qquad$$

The corrected R^2 is then given by

$$R_c^2 (n_m - n_{\text{loop}}^{\text{unc}}) = R_m^2 n_m - R_{\text{loop}}^2 n_{\text{loop}}^{\text{unc}}, \qquad (7)$$

where the index c stands for "corrected" and m for "measured," for the infinite-string ensemble and R_{loop}^2 is the measured value for the loop ensemble. In Table III we list the configurational parameters q and b, together with $n_{\text{loop}}^{\text{unc}}$.

Including the (overcompensating) correction Eq. (7) should give us some idea of the systematic errors, but it corrects almost all the results of Table II for the fractal dimension down by only $D_c \approx D - 0.002$, so that the statistical errors overshadow the systematic ones by far, except for the

TABLE III. Configurational parameters q and b of the loop distribution Eq. (6) for U(1) strings, with the expected number of loops unaccounted for in the measurements. It should be noted that there are systematic computational errors that increase as the number of strings decreases, because the loops need to be grouped in increasingly large length intervals. The errors quoted here are statistical errors only. q is not rescaled by the total string number, to make the extraction of n_{loop}^{unc} more transparent.

Number N of discretization points	q	b	Number of ''infinite'' strings n_m	Maximum length of strings Λ	n_{loop}^{unc}
3	5576 ± 1003	2.530 ± 0.022	6334	50 000	34 ± 19
7	3398 ± 495	2.456 ± 0.018	6394	50 000	54 ± 24
15	4506 ± 628	2.507 ± 0.018	6382	50 000	37 ± 16
31	3930 ± 646	2.492 ± 0.020	6412	50 000	39 ± 20
63	4270 ± 715	2.500 ± 0.020	6346	50 000	38 ± 20
127	3350 ± 637	2.461 ± 0.023	6373	50 000	50 ± 30
255	796 ± 201	2.425 ± 0.035	1903	1 000 000	6 ± 6
255	36800 ± 3200	2.468 ± 0.014	65773	2 000	2240 ± 550

ensemble of 100 000 strings with upper cutoff $\Lambda = 2000$, where there are many miscounted loops, but small statistical errors. In this case, the prediction gets corrected down to $D_c = 2.031 \pm 0.007$. This makes all the measurements of the fractal dimension at different intermediate and long scales just consistent with each other, so that we would need somewhat better statistics than we have accessible at this moment to see whether there is some physical effect or just a conspiracy of statistical fluctuations suggesting a tendency for strings to be slightly self-seeking on intermediate scales (this is indeed implied by a running of the effective fractal dimension, also implying weak violations of scale invariance, as observed for $N=3$ by Bradley *et al.* [28]). However, we have not yet explored the third possible source of errors: the averaging procedure. Table II was obtained by taking $\langle R^2 \rangle$ over the ensemble of strings, i.e., we measured the exponent κ in

$$\frac{1}{N} \sum_i^N R_i^2(l) \propto l^{-\kappa}$$

and defined $D = 2/\kappa$. The question is whether this is the best possible way of defining a fractal dimension, i.e., whether this is a good averaging procedure. However, if the results depend on the specifics of the averaging procedure, then the scaling hypothesis is in trouble because if, for example, the ratio $[\langle R(l)^n \rangle]^{1/n}/\langle R(l) \rangle$ varies with l, then the string network obviously does not look the same on all scales. The ratio of all the moments of the probability distribution for $R(l)$ has to be such that all the $[\langle R(l)^n \rangle]^{1/n}$ stay in a fixed proportion to each other for $l/\xi \gg 1$. In the polymer literature, such ratios are called ''(universal) amplitude ratios.'' It turns out [23] that the fractal dimension for a truly scale-invariant walk is also independent of the definition of $R(l)$ itself, which could be the mean end-to-end distance (which is what we use), the radius of gyration (which is the average separation of *all* point pairs on a walk segment of length l), or the root-mean-square distance of a monomer from the endpoints. So, if the averaging procedure is the reason for the discrepancies in Table II, scaling is noticeably violated up to lengths of several tens of thousands of correlation lengths. The same

argument leads to another important remark: *If scaling is violated, Eq. (5) is not only violated but also ambiguous*, because an unambiguous definition of D requires an unambiguous convention of how the average of R^2 is to be extracted. Such a convention is not necessary if the string network scales.

This makes it very easy to check that the scaling hypothesis is satisfied. First, when we compare the mean values of the measurements for D and b, they satisfy Eq. (5) extremely well. The average b in Table III is 2.4821, whereas $1 + 3/D$ with the average D, 2.0235, is 2.4826. All we need to show now is that the fluctuations in Table II are not systematic. There are two ways of doing this: Either we improve the statistics of the measurement, hoping that the values converge toward each other (presumably somewhere near the range [2.0235,2.0241], which corresponds to the means measured for b and D), or we show that the ratio of $\langle R^2 \rangle$ to $\langle R \rangle^2$ stays fixed. Here we prefer the latter of the two because it will confirm that there are no problems arising from the specific averaging procedure we used, whereas simply increasing the statistics does not give us this reassurance. Strictly speaking, showing that the ration of $\langle R^2 \rangle$ to $\langle R \rangle^2$ stays fixed does not prove scaling unless one shows that *all* the ratios $[\langle R(l)^n \rangle]^{1/n}/\langle R(l) \rangle$ stay fixed for infinite strings.

We did this by reproducing Table I with exactly the same ensembles (i.e., ensembles having the same random number seed), but using $l \propto \langle R \rangle^D$ instead of $l \propto \langle R^2 \rangle^{D/2}$, as is used in all the other measurements. The comparison is shown in Table IV.

It can be seen that the measurements of the fractal dimension agree with each other better than to be expected from statistical errors alone. This indicates that there is not only no measurable discrepancy between the scaling of different moments of the distribution for $R(l)$, but also there are correlations between those moments for any finite string ensemble, so that, unfortunately, one cannot really exploit more than one moment of the distribution to extract two or more statistically *independent* measurements for D from a single ensemble. This is an important observation, as it justifies not only to keep on using the averaging procedure we used from the start, but it tells us that there is no gain of statistical

TABLE IV. Comparison of two different ways of averaging R to obtain the fractal dimension. It can be seen that both lowest moments of the probability distribution yield the same fractal dimension to higher accuracy than expected from the statistical error margins.

Number of strings	Number of ensembles	Average D with statistical error, based on $l \propto \langle R^2(l) \rangle^{D/2}$	Average D with statistical error, based on $l \propto \langle R(l) \rangle^{D}$
1000	10	2.027 ± 0.012	2.027 ± 0.013
500	10	2.022 ± 0.014	2.019 ± 0.014
250	10	2.035 ± 0.016	2.029 ± 0.018
100	10	2.021 ± 0.030	2.027 ± 0.027

accuracy in keeping track of more than one such average. We conclude that the averaging procedure does not introduce additional systematic errors in Table II. We will therefore continue to measure D by fitting $D/2$ in $l \propto \langle R^2 \rangle^{D/2}$ only.

Summing up our analysis of Table II, we conclude that our averaging procedure does not introduce systematic errors, but correcting for the wrong counting of the loops longer than Λ as infinite makes the measurements (just) consistent with each other. Keeping in mind, however, that, for the above explained reasons, this correction is likely to be too generous, we have to agree with the conclusions of [28] that the existence of a slightly scale-dependent fractal dimension has to be accepted as given. This is further supported by the observations in Sec. IVA. Table II also suggests that the very-long-string limit of D may be exactly 2.

C. Results for a perfect $\mathbb{R}P^2$ symmetry

All the qualitative arguments stay the same for an $\mathbb{R}P^2$ symmetry, as it is exhibited, e.g., by nematic liquid crystals [1]. For an $\mathbb{R}P^2$ symmetry the vacuum manifold is a sphere with opposite points identified [$\mathbb{R}P^2$ is therefore identical to S^2/Z_2 or SO(3)/O(2)]. In nematic liquid crystals the occurrence of this symmetry is easily understood: The molecules are mirror-symmetric rods or disks and the ground state of the theory is reached when all rods have the same orientation. If the phase change can propagate faster than the fluctuations in the rotational degrees of freedom of the molecules, this is only achievable locally [29].

We have used only a minimal discretization and a continuous $\mathbb{R}P^2$ group to compare measurements of configurational parameters. The minimal discretization consists of the vertices of an icosahedron embedded in the sphere, as depicted in Fig. 5. The uniqueness of the geodesic rule and the definitions of noncontractible paths on this discretization of $\mathbb{R}P^2$ are both immediately obvious from Fig. 5. Noncontractible paths are those that follow an odd number of those links that cross the equator. Flux conservation is easily established too: Every tetrahedron edge has either one of the broken lines associated with it (i.e., it carries the field values into the other half sphere) or a solid line. Changing any one of the links with respect to this behavior changes the flux in two triangles. Thus the total flux can only be changed in steps of two (or zero) and the number of triangles having strings going through them is always even. By going through the different combinations, it is easy to convince oneself that the thus constructed strings are also self-avoiding, i.e., that no tetrahedron has four faces penetrated by strings.

In the continuous case, the geodesic rule can be realized as follows. Let the random field assignment on a spatial vertex be a random vector of the upper unit half sphere. If the vacuum manifold $\mathcal{M}$ were to be this half sphere only, the length of the geodesic between two points on $\mathcal{M}$ would just be the angle between the two corresponding vectors. If $\mathcal{M} = \mathbb{R}P^2$, the geodesic is therefore either this angle or its complement, whatever is smaller (the probability that a pair of points is connected by a geodesic of length exactly $\pi/2$ is zero). Whenever we need to take the complement of the angle between the two vectors on the upper half sphere, the geodesic will therefore cross the equator. This happens if the two field angles have a dot product smaller than zero. Since there are three vertices to each face of the tetrahedra on our lattice, we need to take all three pairwise dot products. If the curve drawn by the geodesics has crossed the equator an even number of times, then it is contractible, otherwise a string has to pass through the corresponding triangle. Of course, a similar criterion has to be possible for any Z_2 string, and was used for the Z_2 strings appearing in the breaking of SO(3) in [30], using a bounding sphere instead of a bounding circle. If a closed path on the SO(3) manifold
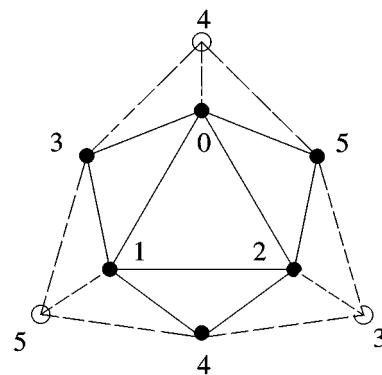

FIG. 5. Minimal discretization of $\mathbb{R}P^2$ and its geodesics. It is obtained by discretizing the points on the sphere by the vertices of an embedded icosahedron. We have to imagine that we look at the icosahedron facing one of its triangles head-on. The ''sphere'' is completed by identifying opposite points. Where necessary to identify all geodesics, points of the lower ''half sphere'' have been drawn, connected by dashed lines. Every point can be connected with any other point by exactly one of the links, so that the geodesic rule is unique. The noncontractible paths are the ones that go along an odd number of broken lines, because broken lines lead onto the other half sphere.

TABLE V. Measurements for the fractal dimension of continuous $\mathbb{R}P^2$ strings.

$\mathbb{R}P^2$ Representation	Number of strings	Cutoff Λ	Fractal dimension D	b
Continuous	3 000	100 000	1.979 ± 0.023	2.59 ± 0.05
	100 000	2 000	1.971 ± 0.001	2.643 ± 0.014
Discrete	10 000	10 000	1.975 ± 0.014	2.62 ± 0.03

crosses the bounding sphere an odd number of times, it is noncontractible. Therefore, if the field values on the vertices of a particular triangle are (in the ''vector on the upper half sphere'' representation) the vectors $\vec{v}_i$, $i = 1, 2, 3$, then the string flux through the appropriate triangle is $n = \frac{1}{2}\{1 - \text{sgn}[(\vec{v}_1 \cdot \vec{v}_2)(\vec{v}_2 \cdot \vec{v}_3)(\vec{v}_1 \cdot \vec{v}_3)]\}$. It cannot have a negative sign because $\mathbb{R}P^2$ strings are nonorientable. This is a direct consequence of the nonorientable nature of the source field: It is apparent that the sum of two noncontractible paths on $\mathbb{R}P^N$ is always contractible as a noncontractible path is one that ends in the antipode of the starting point. The concatenation of two noncontractible paths therefore ends at the starting point itself. This means that any $\mathbb{R}P^N$ string is any other string's antistring in the sense that any two strings (parallel to each other) can form objects that are no longer topologically stabilized. Flux conservation is easily proved [16], but a continuous representation of the $\mathbb{R}P^2$ symmetry suffers from the same uniqueness problems as a U(1) string on a cubic lattice because a single tetrahedron can carry two strings [imagine, for instance, the vectors $(\theta_1, \phi_1) = (0,0)$, $(\theta_2, \phi_2) = (\pi/2 - \epsilon, 0)$, $(\theta_3, \phi_3) = (\pi/2 - \epsilon, 2\pi/3)$, and $(\theta_4, \phi_4) = (\pi/2 - \epsilon, 4\pi/3)$, which for a range of small ϵ has every face penetrated by a string]. To avoid random matching of open string segments (which might introduce an unnatural bias towards Brownian statistics on large scales [6]), we chose to connect the free ends in such a way that in case of ambiguities, every string goes through a pair of faces that share an edge of length a, i.e., the edge length of the bcc lattice. The measured ensembles of $\mathbb{R}P^2$ strings are listed in Table V.

The continuous $\mathbb{R}P^2$ strings do not seem any more Brownian than the ones that are forced to be self-avoiding. This may indicate [as in the case of U(1)] that the discretization of the vacuum manifold does not significantly affect the measurements for perfect symmetry, maybe because we have not allowed random reassignments of string pairs to each other, but we have not checked whether a random solution to the problem of uniqueness would indeed bias the statistics towards Brownian configurations. In any case, no discretizations of $\mathbb{R}P^2$, other than the minimal one, have been investigated at this stage. In fact, no discretization that would be finer than the minimal one, but still force self-avoidance, is known to us. Interestingly enough, a discretization produced by embedding a tetrakaidekahedron into the two-sphere is uniform. Uniform distribution of the lattice points on the sphere is a necessary criterion for unbiased data (cf. the discussion in the following sections). However, it is easy to convince oneself that many of the vector pairs in that scheme are at right angles to each other, introducing ambiguities in the definition of the string flux through a triangle. Another discretization, achieved by embedding a dodecahedron in the sphere, produces a discretization that does not

exhibit these ambiguities, but does allow two strings to penetrate a tetrahedron. The appropriate proof is developed in Appendix A.

We should mention that SO(3) strings have been measured to have a similar tendency to have lower fractal dimension and therefore higher values for b. Kibble [30] arrives at values of $D = 1.950 \pm 0.037$ and $b = 2.546 \pm 0.065$. It is therefore possible that such deviations are generic for either $\mathbb{Z}_2$ strings or for higher-dimensional vacuum manifolds. We will discuss this issue further later.

III. STRING PERCOLATION AND BIASED SYMMETRY BREAKING

A. Low string density

Drawing lessons from polymer statistics, the fact that our algorithm generates nearly Brownian strings could be a result of the dense packing of strings. From what we have measured so far, there is a strong caveat to that statement: The continuous $\mathbb{R}P^2$ strings are actually denser ($1/\pi$ strings per face [25]) than the continuous U(1) strings ($1/4$), but exhibit more ''self-avoiding'' statistics. This trend also holds for the minimally discretized ensembles [$5/18$ for $\mathbb{R}P^2$ and $2/9$ for U(1)]. So how does the string density affect string statistics?

We have already shown that for minimally discretized U(1) strings, a Hagedorn-like transition [31–33] occurs below a critical string density [6]. According to Vachaspati [25], we can achieve variations in the string density by inducing correlations in the order parameter by lifting the degeneracy of the manifold of equilibrium states. This reduces the probability of a string penetrating the face of a lattice (thus we can generate an ensemble with the *average* string density fixed at will; physically, one can think of this as applying an external field, which spoils the symmetry of possible ground states), but *increases* the dimension D, which argues against the identification of strings with polymers. There is a critical density below which there are no infinite strings. In the low-density phase there is a scale c that appears in the loop length distribution

$$dn = al^{-b}e^{-cl}dl \tag{8}$$

as a cutoff. As the critical density is approached from below, $c \rightarrow 0$ and the mean square fluctuation in the loop length

$$S = \langle l^2 \rangle - \langle l \rangle^2$$

diverges (see exponents γ and ψ in Table VII).

This divergence signals a phase transition, in some ways analogous to the Hagedorn transition in relativistic string theory at finite temperature. This has been implicated in many branches of physics. Previous studies [34] deal with

string dynamics and can treat the ensemble in thermal equilibrium. The ensemble in [34] is therefore very appropriate for situations where the critical temperature is approached slowly. Vachaspati's algorithm enables us to measure directly the string statistics such as the critical density, the dimension, and the critical exponents and to test the validity of the hypothesis of scale invariance for the *initial conditions*, which cannot be expected to be thermalized.

B. Low string density and the Hagedorn transition

From the ''microscopic'' point of view, Vachaspati [25] argued for such a Hagedorn-type transition to occur at low string densities with the following reasoning. Consider a string formation simulation on a cubic lattice. The probability of a string passing through a certain face 1 of the cell is p_s. Since the plaquette opposite face 1 is causally disconnected, the probability for it to have a string passing through it is also p_s, regardless of the actual situation at face 1. Therefore, the probability for a string to bend after entering a cell is $1 - p_s$. Now, if we reduce p_s, the bending probability increases and the chances of the string closing up to form a loop also increases. As Vachaspati argues, ''This tells us that by reducing the probability of string formation, or equivalently, by decreasing the string density, we can decrease the infinite string density and increase the loop density.''

Vachaspati then goes on to construct a model with $\mathbb{Z}_2$ strings (i.e., nonorientable strings), in which he assigns either $+1$ with probability $1 - p$ or -1 with probability p to each link of the lattice (on a periodic lattice). A string is said to pass through a plaquette if the product of the field values on the associated links is -1. Although reminiscent of it, this is not quite identical to the way we constructed our $\mathbb{R}P^2$ strings in Sec. III A, because whether a $+1$ or a -1 is ''assigned'' to a link in the continuous $\mathbb{R}P^2$ case depends on the relative angles between the three vectors involved, so that the assignments to the links are not entirely uncorrelated. If they were, then the probability of an $\mathbb{R}P^2$ string passing through a triangular (or in fact any) plaquette should be $\frac{1}{2}$, whereas it is (for a triangular plaquette and continuous $\mathbb{R}P^2$) $1/\pi$ [25]. The way strings are constructed in Ref. [25], however, is appropriate to model $\mathbb{R}P^\infty$ strings. We can see this by the following argument. It is well known that the different components of a random unit vector in $\mathbb{R}^N$, in the limit $N \rightarrow \infty$, become mutually uncorrelated Gaussian random variables with standard deviation $\sigma = \sqrt{N}$. The inner product of any two random unit vectors therefore will have a positive or negative sign with equal probability. The relative angles to a third unit vector and in particular their signs are then completely uncorrelated to this angle, so that taking the sign of the product of uncorrelated Gaussian random variables would indicate whether or not a sequence of geodesics between random points on $S^\infty/\mathbb{Z}_2$ will cross the horizon.

With this model Vachaspati observed that, as the symmetry bias is increased, a lot of string mass is transferred from infinite strings to loops, so that the loop density actually increases. This is not what one would expect, e.g., from statistical arguments for a box of (noninteracting) strings in equilibrium [33], so that one should not assume *a priori* that the string statistics right at the phase transition will follow statistical-mechanics arguments. Another prediction of statistical-mechanics arguments is that, at low densities, the loop distribution is described by Eq. (8), with $b = \frac{5}{2}$. Vachaspati, however, measures values consistent with 2 (within large statistical errors). As one increases the string density again, approaching the scaling regime, c approaches zero from above, signaling a phase transition (c can be interpreted as the inverse of some characteristic length scale arising from the breakdown of scale invariance).

Vachaspati's argument, relating the probability of a string forming at a particular lattice plaquette to the Hagedorn transition, actually does not go far enough. It supports the notion that the string is getting wigglier as we decrease the density, which could, strictly speaking, result in just a rescaling of some of the parameters, but none of the exponents: The scaling function in Eq. (2) could converge towards a different constant, the factor q in Eq. (6) could change, all without changing D or b, which determine the global properties of the network after the local properties have been absorbed into appropriate prefactors. In particular, the *complete* disappearance of infinite strings is not explained convincingly. What Vachaspati observes in Monte Carlo measurements, however, *can* be explained on the microscopic level. Vachaspati varies the string density by decreasing p, the probability for a link to have the value -1 assigned to it. Let us take it to the extreme and assume that all the links that have -1 assigned to them are so rare that they are usually isolated from each other, submerged into a sea of links with field value $+1$. Then it is obvious that a string loop of minimal length winds around each of these links, so that there will be nothing but a few isolated short string loops. We can take it further and ask ourselves what happens when two such links are adjacent to each other. If they are consecutive links with the same orientation, they will have their own loops of length 4; if they are in different spatial orientations, a loop of length 6 will form, as depicted in Fig. 6. This figure also illustrates that, because the length of the strings seems to be intimately linked to the size of the -1-link clusters, *in the Vachaspati model, the Hagedorn transition is almost a bond percolation problem*, except that parallel bonds touching each other (i.e., bonds along the same line) do not connect their strings with each other and parallel bonds that are just one lattice spacing apart do. There are more configurations of these -1 links that break this correspondence between the Vachaspati model and bond percolation (e.g., a flat cross of four -1-links produces two separate loops). Thus, although there is no one-to-one correspondence, one still intuitively expects the Hagedorn threshold to be close to the bond percolation threshold. Indeed, Vachaspati measures a percolation threshold of $p_c \approx 0.29$, while the threshold for bond percolation on a simple cubic lattice in three dimensions is $p_c = 0.3116$ [35]. There is more to be learned from the correspondence of the Vachaspati model with bond percolation. To get a respectable number of large but isolated lattice animals, we have to approach the percolation threshold from below. At the threshold, the percolating cluster has a well-defined fractal dimension. Thus we conclude that scaling must be restored as the percolation threshold is approached from below and a fractal dimension will begin to become well defined. Finally, we shall just mention that one can easily derive the general form of Eq. (8) by similar percolation arguments.
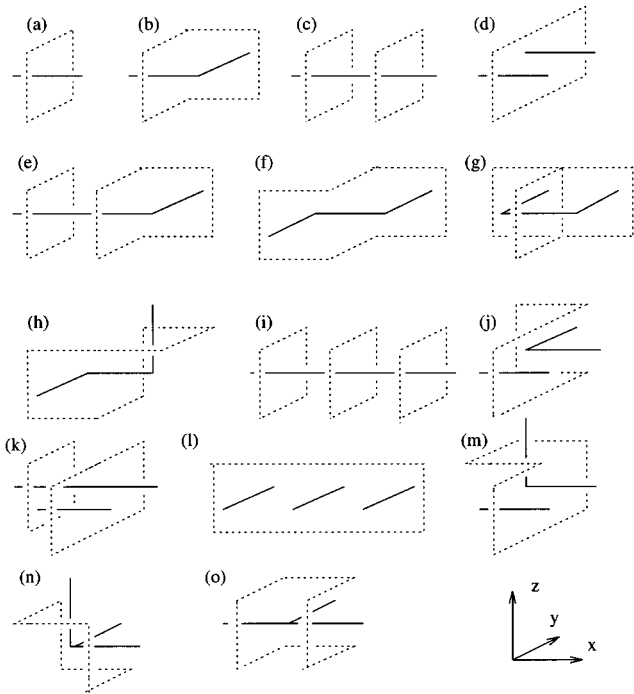
FIG. 6. String loops formed in this model, with one isolated link with field value -1 (a) and with two [(b)–(d)] or three [(e)–(o)] such links adjacent to each other. The fact that infinite strings disappear looks very reminiscent of a bond percolation problem for the ''(-1) links,'' except that two consecutive strings, if they are aligned, do not surround themselves with pieces of the same string [the prototype is (c), other examples are (e), (i), and (k)], whereas neighboring parallel links do [the prototype is now (d), with other examples being (j)–(m)].

Not only can we now claim to understand the microscopic aspects of the lattice description of this Hagedorn-like transition, but we also expect this transition to have many properties of a percolation transition. We can relate many variables and critical exponents of the Hagedorn transition to critical behavior in standard percolation transitions. With his model, Vachaspati got qualitative indications of many of the results that are to follow here. With the infinite-lattice and the hash-table algorithms used in [6] and presented in [16,7] we have some advantage when extracting numerical data or attempting reasonably large ensembles for good statistics.

C. String percolation in the Vachaspati-Vilenkin algorithm

Now we need to go back to the more realistic model: the Vachaspati-Vilenkin method on a tetrahedral lattice. The string density can be varied (once the lattice is chosen) only by lifting the degeneracy in the vacuum states, i.e., by making some vacuum states less likely than others. Once the details of the discretization of space and of the vacuum manifold are chosen, *the initial string density, and in particular the density in infinite strings, can be changed only by spoiling the vacuum symmetry*. There has been recent work on dynamical scaling [5]. Dynamical scaling is quite different from scale invariance and is exhibited if the system looks statistically the same at all times, on length scales that may vary with time according to some power law or some other function of time. This does not imply that the system is scale

invariant. In fact, in a *scale-invariant* system (if it stays scale invariant), dynamical scaling is a misplaced concept because there is no length scale that could evolve in time. Parameters that are *not* scale invariant, and whose dynamical scaling it therefore makes sense to observe, e.g., the average string-string separation, are those parameters that are affected by lattice effects in the VV algorithm. The work in Ref. [5] implies that the ratio of the densities in string loops and in infinite strings may be freely variable, based on the realization that this ratio depends on the lattice description invoked. We should stress that the density in loops depends on the lattice. Our tetrahedral lattice allows smaller loops (in units of correlation lengths) than a cubic lattice, and one expects more loops to appear because the low cutoff in Eq. (4) gets shifted to lower values. There is also a difference in this ratio depending on the discretization of the vacuum manifold and on the vacuum manifold itself. Thus, whereas we agree with the general argument of Ref. [5], we will show in Sec. VIA that there is probably a lower limit to the amount of infinite string that has to appear and infinite strings would therefore be a generic feature of the VV algorithm on a regular lattice. This issue is still controversial, but in some simple cases, such as the Vachaspati model, we can develop a percolation theory understanding for the emergence of an infinite string network. Had Vachaspati used a tetrahedral lattice, there would still be infinite strings, as the bond percolation problem threshold for the bcc lattice is $p_c = 0.1803$ and the symmetric case has $p = 0.5 > p_c$. The reason why the bond percolation threshold is reduced compared to the simple cubic lattice is, from the percolation theory viewpoint, that there are more bonds per lattice site. Within the string network picture, the reason is that we have a finer mesh and therefore a higher string density. In fact, on any three-dimensional lattice $p_c < 0.5$, so that the appearance of infinite strings is lattice independent. Serious lattice ambiguities would arise only if strings (under the same physical conditions) percolate on one lattice, but not on another, i.e., if p_c lies in between percolation thresholds of different lattices. No such model is known.

Because of its better correspondence to a physical situation, let us consider another brief example, taken from the measurements in Sec. IV. Take the tetrahedral lattice with a minimal discretization of U(1). Let us denote the three possible field values by 0, 1, and 2. We introduce a bias in the symmetry such that the value 2 is assigned with the probability p_s and the other two values have the probabilities $(1 - p_s)/2$. Without loss of generality, let us constrain ourselves to biases with $p_s \leq \frac{1}{3}$. We can produce infinite strings only if all three field values percolate. In particular, this implies that $p_s > p_c$, where the critical value p_c is the site percolation threshold of a bcc lattice, $p_c \approx 0.246$ [35]. In the unbiased case $p_s = \frac{1}{3}$. Again, this is higher than the site percolation on *any* sensible lattice such that the appearance of infinite strings is a generic feature. From measurements in Sec. IV, we deduce $p_c = 0.2476 \pm 0.0014$. The agreement is almost suspiciously good, but certainly justifies the percolation theory arguments for an intuitive understanding of the Hagedorn transition. Had we taken a simple cubic lattice, we would still be above the percolation threshold $p_c = 0.3116$ and get infinite strings [17]. In both Vachaspati's $\mathbb{Z}_2$ model and the minimally discretized U(1) model we get infinite

strings irrespective of the lattice we are using. To be precise, a diamond lattice would not allow site percolation for $p = \frac{1}{3}$. However, because it has hexagonal ''plaquettes'' (the quotation marks are to indicate that the plaquettes are not planar), it is unsuitable not only for a simulation of the Kibble mechanism, but also for the percolation theory argument developed here. This is because two consecutive plaquettes are not everywhere connected by link walks of length one, so that the lattice points with disfavored vacuum values do not need to neighbor each other directly to allow strings to percolate and site percolation with *next-nearest* neighbors should be our reference in this case. Percolation phenomena have long been known to be independent of the microscopic details of the lattice. This may lend some support to the assumption that, in Vachaspati's $\mathbb{Z}_2$ model (and maybe more generally for $\mathbb{Z}_2$ strings) and for U(1) strings the emergence of a network of infinite strings is a generic feature. Although the correspondence of the Hagedorn transition to a percolation phenomenon seems rather strong, we suffer from the same deficiency here as most of percolation theory does: There is no analytic proof.

In many respects, the best we can hope for is to establish a better understanding through better and more numerical measurements. The next section is therefore dedicated to the results of various measurables of the percolation transition. We will present more arguments for the correspondence of the string ensembles with a percolation theory picture later when we discuss the results of those measurements.

IV. NUMERICAL RESULTS FOR BIASED STRING FORMATION

A. U(1) strings

The following convention has been used to introduce a bias for U(1) strings. We discretized the U(1) manifold by $N = 2^n - 1$ points and assigned the following probabilities to each of these points $m \in \{0, 1, 2, \ldots, N-1\}$:

$$p(m) = C^{-1} \exp[-\eta \cos(2\pi m/N)],$$

where η is the bias parameter and C simply normalizes the probabilities

$$C = \sum_{m=0}^{N-1} \exp[-\eta \cos(2\pi m/N)]. \tag{9}$$

Unless stated otherwise, we will quote results from the minimal discretization of U(1) in this section, i.e., $N = 3$. The reason why this is the best-studied ensemble is that it corresponds most closely to a site percolation problem and therefore relates best to the discussion of the results in Sec. V. First, we confirm that Eq. (8) gives an extremely good fit for the loop distribution beyond the percolation threshold. Typical such fits are shown in Fig. 7. In Fig. 8 we compare the loop distributions for different biases. It can be seen that for low bias the string density in loops increases with increasing bias, in agreement with Vachaspati's measurements. The density in loops as a function of η is shown in Fig. 9. The mass density in loops (in units of segments per tetrahedron) is obtained in the following way. Let p_t be the probability for a triangle to carry a string segment. Since the number of

triangular plaquettes on a tetrahedral lattice is twice the number of tetrahedra, the average number of strings per tetrahedron is then $2p_t$. Whenever we start tracing a string, we start at a randomly chosen string segment (out of all possible segments on the imagined infinite lattice). Thus we will pick elements belonging to an infinite string according to their density ratios, such that

$$\rho_{\text{loop}} = \rho_{\text{total}} \frac{N_{\text{loop}}}{N_{\text{total}}}, \tag{10}$$

where N_{total} is the number of strings we have traced in the ensemble and N_{loop} is the number of those that turned out to be loops. The unknown parameter is p_t. For the minimally discretized U(1) manifold, the sum C of Eq. (9) is just

$$C = 2 e^{\eta/2} + e^{-\eta}$$

and the probability for a triangle to exhibit all three vacuum values on its vertices reduces to the very simple form

$$p_t = \frac{3!}{C^3}. \tag{11}$$

For other discretizations of U(1) we use exact numerical summations to extract p_t, as the functional form becomes a rather complicated sum of a number of terms that increases as N^3 and has to be evaluated for all values of η used in the data set. It turns out that the defect density increases slightly when a finer discretization of the vacuum manifold is used. This tendency is also observed in Monte Carlo simulations of texture formation [21] and monopole formation [15] and in the $\mathbb{R}P^2$-string measurements in Sec. IV B. The string density per tetrahedron in terms of the bias, for the minimal discretization of U(1), is therefore given by

$$\rho_{\text{total}} = \frac{2 \times 3!}{(2e^{\eta/2} + e^{-\eta})^3}.$$

This can be used as a reparametrization of the bias, so that all variables X scaling like $X \propto |\eta - \eta^\star|^\chi$ near the critical point will also scale as $X \propto |\rho_{\text{total}} - \rho_{\text{total}}^\star|^\chi$, as it is a smooth and analytic function of the bias. Note that, e.g., the mass density in loops cannot be taken as such a reparametrization, as it is not smooth at the critical point. Figure 9 shows the separate mass densities in infinite strings and loops. Note that the energy in loops at the percolation transition still exceeds the energy in infinite string at zero bias. In analogy with the polymer literature, we could say that this transition (when approached from the nonpercolating phase) is very efficient in pumping energy into the entropy terms, i.e., in utilizing new degrees of freedom as the bias is lowered. This is why we call it a Hagedorn-like transition: The Hagedorn transition [31] is associated with an exponential increase in the degrees of freedom, such that (in the thermal situation) the Hagedorn temperature is not reachable, as all the energy, pumped into the system to further increase the temperature, goes into entropic terms of the Helmholtz free energy. However, since our model does not deal with a thermalized ensemble (or with any dynamics at all) we can still reach domains beyond this Hagedorn-like transition.
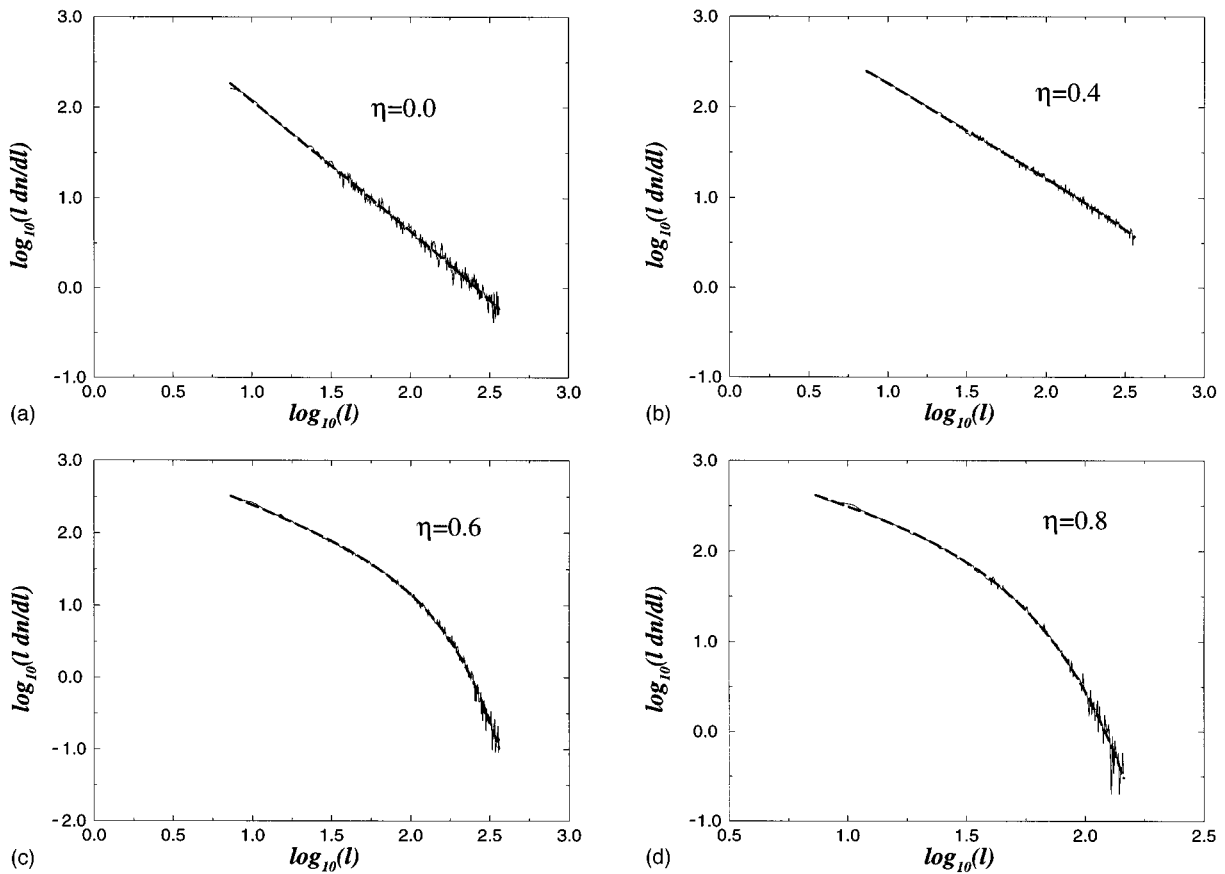
FIG. 7. Typical fits to Eq. (8), with different values of the bias. Where infinite strings are present, no cutoff can be identified. Where $\Lambda < 1/c$ (i.e., for $\eta = 0.4$), the cutoff cannot be seen, as the strings in this ensemble are too short. For higher values of the bias, the cutoff can be recognized clearly. The dashed lines are the fits using Eq. (8). No deviations from a behavior of the type in Eq. (8) can be recognized. The ensemble consists of 100 000 strings with cutoff $\Lambda = 2000$.

Figure 9 allows us to measure the location of the percolation threshold by fitting a power law of the form

$$\rho_\infty \propto (\eta^\star - \eta)^\beta.$$

(Where appropriate, the exponents are named according to their use in percolation theory. In percolation theory β is the critical exponent associated with the strength of the infinite network. Since the total mass density is a smooth function of η, β is also associated with the mass density in loops.) We do this by trying different fixed values of $\eta^\star$ and taking the one that gives the smallest sum of residuals on a log-log
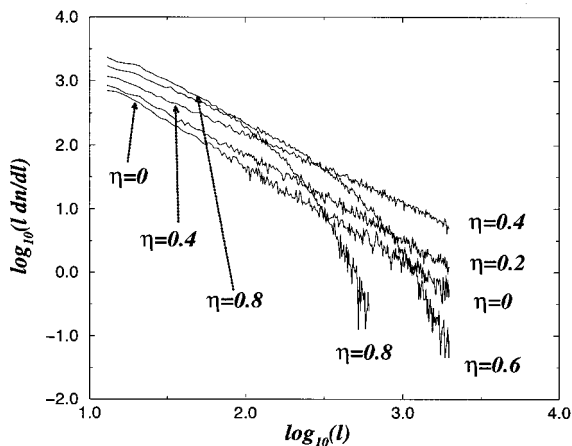
FIG. 8. Comparison of the loop distributions for different bias. It can be seen that the loop density is increased for small bias. For large bias, the running of the cutoff length can be observed.
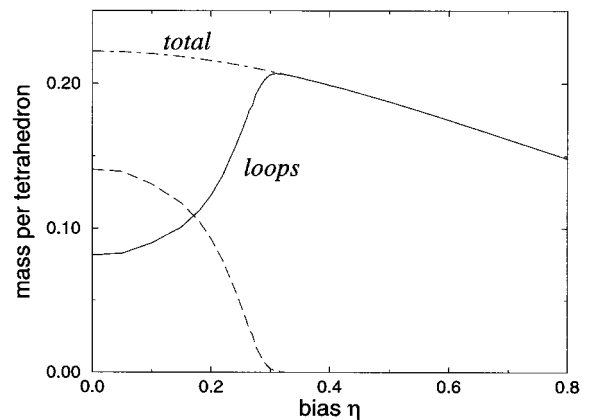
FIG. 9. Mass density in string segments belonging to loops (in units of one per tetrahedron) (solid line), in infinite strings (dashed line), and the total mass density, given by Eqs. (10) and (11). The loop density increases as we approach the percolation threshold from above, and energy from infinite string is transferred into the loop ensemble.
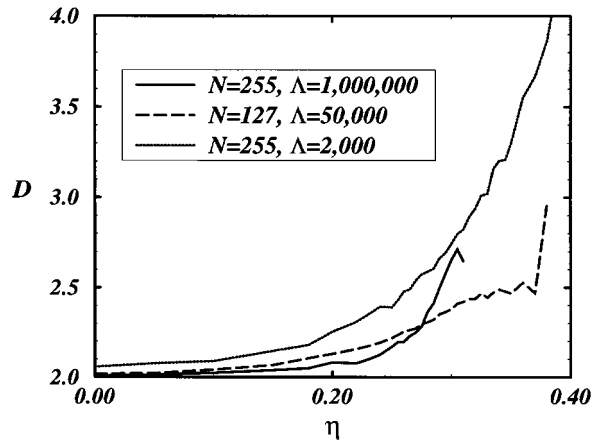
FIG. 10. Fractal dimension of the infinite-string ensemble as a function of the bias, plotted for different values of the upper length cutoff Λ. The measurements are for nearly continuous U(1).
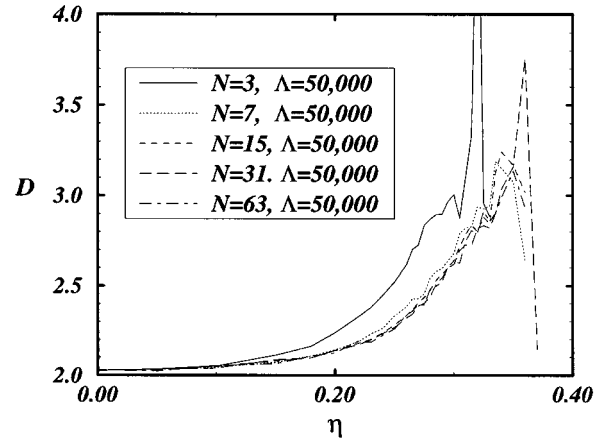

FIG. 11. Fractal dimension of the infinite-string ensemble as a function of the bias parameter, plotted for different discretizations of U(1). The critical biases can be found in Table VII and in Fig. 17.

least-squares fit. This means that the statistical errors are obtained in a less accurate procedure than in Sec. III: Instead of taking many different ensembles, we take the fluctuations of $\eta^\star$ to be such that the sum of the error squares in the linear fit to the plot of $\log(\rho_{\text{loop}})$ vs $\log(\eta^\star - \eta)$ is allowed to fluctuate by a factor 2 around its minimum. The respective slopes will usually differ by an amount of the order of the statistical error. Since we have to vary both a lower and an upper cutoff, as well as the estimate for $\eta^\star$, when searching for the best fit, this method reduces the large computational effort that would be involved if we had extracted statistical errors by measuring many different ensembles for every symmetry group. Quite often, we get very large estimated errors in $\eta^\star$ and the critical exponents because of the many free variables involved. A proper analysis of corrections to scaling, as done in Ref. [28] for the minimally discretized U(1) strings, is necessary, but will be done elsewhere. We measure

$$\beta = 0.54 \pm 0.10,$$

where the fit has been done in the region $\eta \in [0.22, 0.265]$, and the errors are associated with the uncertainty in $\eta^\star$, which is measured to give the best fit at $\eta^\star = 0.279 \pm 0.005$. If the uncertainty in $\eta^\star$ is large, the errors for β quickly get out of hand. The criterion of whether or not one gets a good fit is not very efficient in finding $\eta^\star$, but it is the best we can do. In percolation theory there is a useful procedure (cf. Ref. [35], p. 72) that involves observing how the probability to find a lattice-spanning cluster, as a function of p, scales with the size of the lattice. For example, one could look at how the point where this probability is $\frac{1}{2}$ scales with the system size and then extrapolate where this point will end up as the lattice size goes to infinity. This gives a very good estimate for the percolation threshold only if one keeps track of all clusters generated on a given lattice. We only trace one string at a time, not worrying about the rest of the lattice, so that this method of identifying the percolation threshold does not work.

The fractal dimension is plotted in Fig. 10 for nearly continuous U(1), but with different upper cutoffs Λ, and in Fig. 11 for $\Lambda = 50\,000$, but with different discretizations of U(1).

Five features are noteworthy.

(i) The statistical errors increase with increasing bias because the number of infinite strings in the ensemble becomes smaller.

(ii) The fractal dimension stays nearly constant for very small bias. We observe that the Λ dependence of D as measured in Sec. III is *not* a statistical fluctuation.

(iii) The fractal dimension not only becomes hard to measure near the percolation threshold, but also becomes ill defined beyond it because we are counting many strings wrongly as infinite ($\langle l_{\text{loop}} \rangle$ diverges at the critical point), whereas they will eventually turn back onto themselves and form loops.

(iv) The behavior is largely independent of the particular discretization used, except for the obvious shift in $\eta^\star$, pronounced only for $N = 3$.

(v) The measurements are consistent with a possible assumption that, as $\Lambda \to \infty$, $D = 2$ right up to the critical point.

Measurements for the average loop size—note that we mean the average size of a loop that a randomly chosen string segment belongs to—are shown in Figs. 12 and 13. In Fig. 13 the effects of a finite cutoff are also explained. The behavior is just as one would expect from percolation theory:
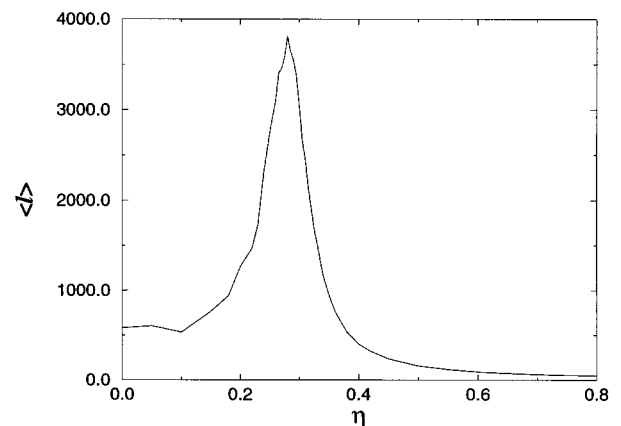

FIG. 12. Average length of a loop in the minimal discretization of U(1), as a function of the bias η. Here $\Lambda = 50\,000$.
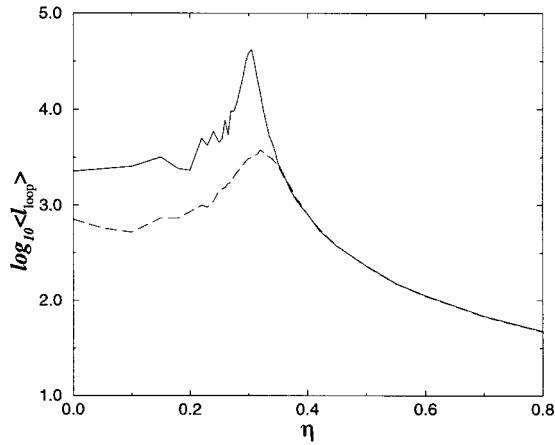
FIG. 13. Logarithm of the average length of a loop [with U(1) nearly continuous: $N=255,127$] as a function of the bias η, for different cutoffs. For the solid line $\Lambda=1\,000\,000$, whereas for the dashed line it is $\Lambda=50\,000$. They should (to a very good approximation) have the same critical bias. However, the lower-cutoff peak has an inflection point, where it deviates from the power-law behavior, well before the critical point (at around $\eta=0.36$), where we need to stop the fits to measure the critical exponents. This effect gets worse with smaller cutoffs and gives rise to inaccuracies in estimating the exact position of $\eta^{\star}$.

In the nonpercolating phase, the main contribution comes from gradually larger clusters as we approach the percolation threshold.

Let us assume the average loop size near the percolation threshold scales as

$$\langle l_{\text{loop}} \rangle \propto |\eta - \eta^{\star}|^{-\gamma}.$$

The best power-law fits to the loop size give

$$\eta^{\star}_{(N=3)} = 0.279 \pm 0.004, \quad \gamma = 1.59 \pm 0.10,$$

measured in the range $\eta \in [0.33, 0.6]$. Again, large errors are associated with the uncertainty of where exactly the percolation threshold lies. This again amounts to a problem with finite-size effects: Unless $\langle l_{\text{loop}} \rangle \ll \Lambda$, the ensemble average will miss out on large contributions from loops wrongly counted as infinite strings.

For bias values below the percolation threshold there is no critical exponent. In this domain the average loop length is a divergent function of the upper cutoff and an average length becomes ill defined. This is obvious from Eqs. (4) and (5) and the fact that $\langle l_{\text{loop}} \rangle \propto \int_{\lambda}^{\Lambda} l^{-b+2} dl$ while $D > 3/2$. This problem is alleviated in the nonpercolating regime (where the loop distribution is exponentially suppressed by an additional factor of e^{-cl}), as long as $\xi = 1/c \ll \Lambda$, i.e., for values of η not too close to the critical bias. The same argument holds for any higher moment $\langle l_{\text{loop}}^{n} \rangle$ of the loop distribution.

Another way of investigating the ensemble is by means of a partition function, which is the (unnormalized) sum over probabilities $p(l)$
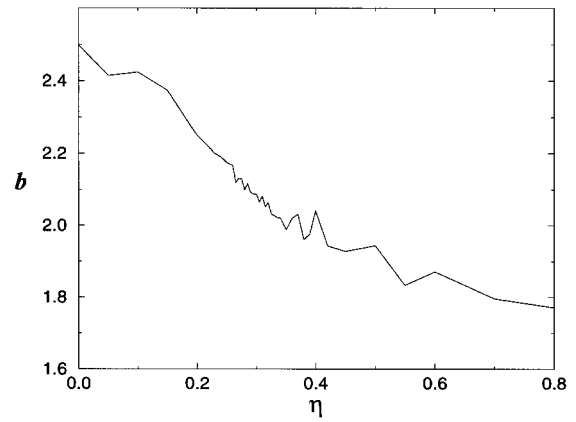
$$Z = \sum_{l} l^{-b+1} e^{-cl}. \tag{12}$$

FIG. 14. Loop distribution exponent b in Eq. (8), for an ensemble of 10 000 strings with cutoff $\Lambda = 50\,000$.

This is not a thermal partition function, but should rather be viewed as a generating function for the moments of the loop size distribution. If we wanted to use thermodynamics language, then the factor l^{-b+1} would be proportional to the density of states for given l (or one could say it is a suitably defined integration measure, which amounts to the same) and $\xi = 1/c$ fulfills the role of a temperature, as is shown in Appendix C. This only serves to show that a thermodynamic nomenclature is inappropriate, as the threshold for a dynamical Hagedorn transition really lies at a finite temperature, and instead of having the temperature diverge at the critical point, one would have to factor out the divergent terms and pull them into the density of states. This distinction becomes meaningless in our nonthermal ensemble. Our ''partition function'' can equally well be viewed as a sum over the density of states only, with a critical temperature (or density) dependence. Although these names are slightly inappropriate, they give the right behavior, e.g., for the average ''energy'' $\langle l_{\text{loop}} \rangle$ in the nonpercolating phase

$$\langle l_{\text{loop}} \rangle = -Z^{-1} \frac{d}{dc} \ln Z.$$

Since the parameters b and c are crucial in this interpretation, measurements for them are shown in Figs. 14 and 15.
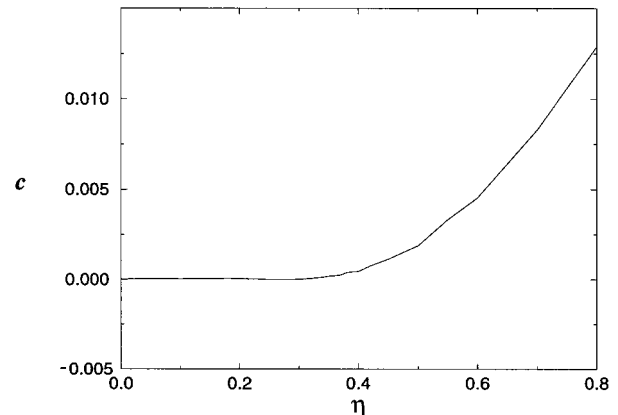

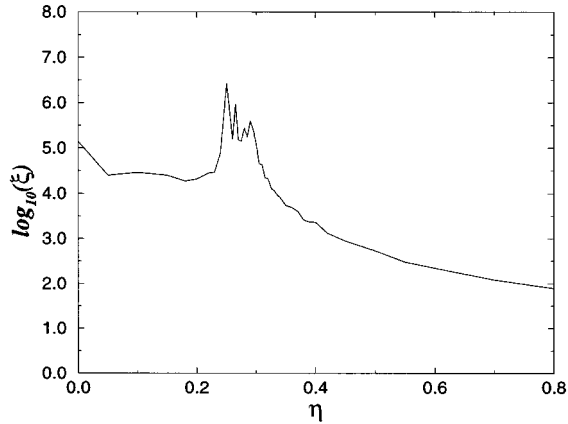FIG. 15. Parameter c in Eq. (8) for the same ensemble as in Fig. 14.

FIG. 16. Parameter $\xi = 1/c$ in Eq. (8) for the same ensemble as in Fig. 14. Very large values in the region $\eta < \eta^\star$ just correspond to statistical fluctuations of c around zero.

Although the parameters b and c are comparably easy to measure, since one does not have to guess $\eta^\star$, there are still large statistical fluctuations for b. For $\eta > \eta^\star$, b is allowed to be smaller than 2, since the scaling relation Eq. (5) no longer holds. It becomes more difficult to measure b at large bias, but it is also less important there since the exponential cutoff dominates the loop size distribution. The diverging length scale associated with the phase transition is $\xi = 1/c$, which is plotted in Fig. 16.

There is another critical exponent associated with the divergence of ξ:

$$\xi \propto (\eta - \eta^\star)^{-1/\sigma}.$$

Implicitly, we measure for the percolation threshold $\eta^\star = 0.280 \pm 0.003$. The critical exponent is

$$\sigma = 0.46 \pm 0.02.$$

Before concluding this section, we measure the critical exponent in

$$\langle l_{\text{loop}}^2 \rangle \propto (\eta - \eta^\star)^{-\psi},$$

which gives (as well as $\eta^\star = 0.277 \pm 0.004$)

$$\psi = 3.79 \pm 0.14.$$

This is the exponent associated with what we called "susceptibility" in Ref. [6], $S = \langle l_{\text{loop}}^2 \rangle - \langle l_{\text{loop}} \rangle^2$, since the diverging behavior of $\langle l_{\text{loop}}^2 \rangle$ dominates in the expression for S because $\psi > 2\gamma$. In terms of the partition function, it is

$$\langle l_{\text{loop}}^2 \rangle = Z^{-1} \frac{d^2}{dc^2} Z$$

such that $\langle l_{\text{loop}}^2 \rangle = -Z^{-1}(d/dc)(Z\langle l_{\text{loop}} \rangle)$. When compared to a Hagedorn transition, however, we want the derivative of $\langle l_{\text{loop}} \rangle$ with respect to a temperature (i.e., any smooth reparametrization of the bias) such that a proper definition of the susceptibility as the energy cost per loop associated with decreasing η would simply diverge with an exponent $(-\gamma - 1)$ [from $\langle l_{\text{loop}} \rangle \propto (\eta - \eta^\star)^{-\gamma}$ and the "energy cost" per loop associated with decreasing the bias becomes

TABLE VI. Summary of the definition of the critical exponents.

$\xi = 1/c \propto (\eta - \eta^\star)^{-1/\sigma}$
$\langle l_{\text{loop}} \rangle \propto (\eta - \eta^\star)^{-\gamma}$
$\langle l_{\text{loop}}^2 \rangle \propto (\eta - \eta^\star)^{-\psi}$
$\langle \rho_\infty \rangle \propto (\eta^\star - \eta)^\beta$

$(d/d\eta)\langle l_{\text{loop}} \rangle]$. Another reasonable definition of a susceptibility would be to define it (at least for $\eta > \eta^\star$) as $(d/d\eta)\rho_{\text{loop}}$, which then diverges with the exponent $(-\beta - 1)$.

This concludes the discussion of the critical behavior of the ensemble with a minimal discretization of U(1). Measurements for the other ensembles with gradually finer discretizations of the vacuum manifold are listed in Table VII. Note that the measurements supercede the ones in [6], where the string ensembles were smaller and fewer compared to the ones available now. A reminder of which exponent is associated with which variable can be found in Table VI.

Ideally, we should have two ensembles for every discretization, one with a small Λ, allowing us to trace a larger number of strings to extract the exponent c, and one with very large Λ, to measure the divergences in the average loop size, to which the loops of length $l \lesssim \Lambda$ contribute the most, as we approach the critical point. However, we do not notice too much of a reduction of statistical errors by doing this (as is done for $N = 255$), so these ensembles have not been generated. A cutoff of $\Lambda = 2000$ is clearly too low to give sensible answers: The very long string ensemble agrees with the medium-cutoff ensembles, whereas the ensemble with cutoff $\Lambda = 2000$ allows hardly any measurements and the one exponent that is measurable is obviously affected by severe systematic errors. This also means that on a finite lattice of about 45^3 lattice points, which amounts to a similar cutoff, measurements like the ones presented here are virtually impossible. In this way we can understand that—for lack of an infinite-lattice formalism of the kind we are using for these ensembles—no one, to our knowledge, has picked up on Vachaspati's model to investigate the Hagedorn transition more closely.

Another consequence of finite cutoff effects is that the measurements of γ give the best fits to a power law and are therefore the best indicators of where the percolation threshold lies. The reason for this is simple to understand: Large loops give the main contribution towards the length average and finite cutoff effects will eventually flatten the peak, which should diverge at the critical bias. However, the contribution of large loops to $\langle l_{\text{loop}}^2 \rangle$ is even more important and finite cutoff effects will spoil the power-law approach to the critical point at a much earlier stage, such that ψ has much larger errors associated with it, and it is recommendable to use estimates for $\eta^\star$ that were obtained from fits of the average loop size to a power-law behavior. The same is true for c (and its associated exponent σ) for similar reasons, which, in an intuitive fashion, one might simply sum up as follows: The smaller the modulus of the critical exponent, the closer one can go towards the critical point without "feeling" finite-size effects.

In Fig. 17 we present the values of the critical string density as a function of the discretization of U(1). It can be seen
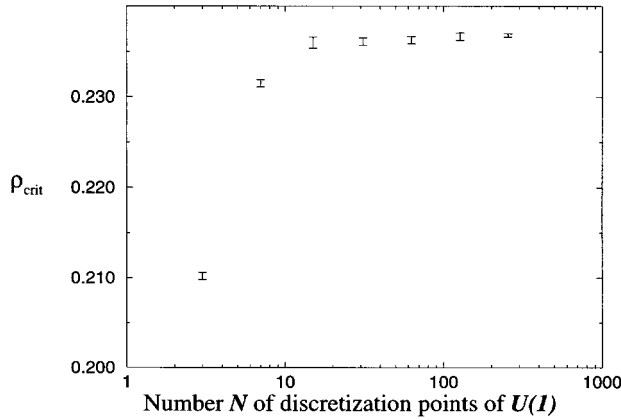
FIG. 17. Critical string density (in units of string segments per tetrahedron) plotted against the number of discretization points for the U(1) manifold.

FIG. 18. Testing the validity of Eq. (5) on a U(1) string ensemble with $N=63$.

that $N \gtrsim 15$ is already very close to the continuous U(1) limit. In this sense, the ensembles with $N \geq 31$ in Table VII can be interpreted as the same measurements over several ensembles of the same kind.

Finally, it is worthwhile testing the validity of the scaling relation for configurational exponents [Eq. (5)], which, as we have discussed before, implies that on intermediate scales loops show the same fractal behavior as infinite strings. Scaling seems to hold roughly. Considering that the exponent b is usually taken from quite noisy data (except when close to the percolation threshold), the agreement seems very good and is shown in Fig. 18 for the $N=63$ ensemble. When c is constrained to be zero, as it was done to obtain Fig. 18, such plots give a very good first estimate of the percolation threshold and the Fisher exponent τ, i.e., the value of b at criticality, which plays a prominent role in Sec. V, when we will come to use the partition function to describe general scaling relations.

B. $\mathbb{R}P^2$ strings

For $\mathbb{R}P^2$ strings our conventions of what the bias means depend on the discretization. For the minimally discretized $\mathbb{R}P^2$ strings (cf. Fig. 5), we define a bias ζ as
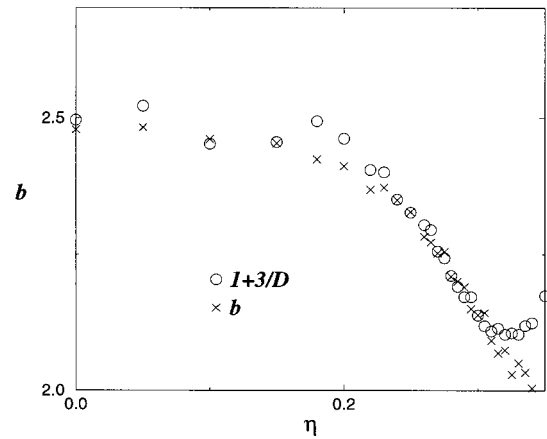
$$p(0) = p(1) = p(2) = \frac{1+\zeta}{6},$$

$$p(3) = p(4) = p(5) = \frac{1-\zeta}{6},$$

such that $\zeta = 1$ corresponds to $\rho_{\text{total}} = 0$, $\zeta = 0$ is a perfect $\mathbb{R}P^2$ symmetry, and $\zeta = -1$ is a perfect U(1) [if all the vacuum values live on the equator, we have the symmetry $U(1)/\mathbb{Z}_2$, which is homeomorphic to U(1) and produces exactly the same results as the minimally discretized U(1) ensembles]. For the continuous $\mathbb{R}P^2$ simulations, we define a bias μ such that the normalized probability density for the polar angle $\theta \in [0, \pi/2]$ (i.e., constrained to the upper half sphere) is

$$dp(\theta) = \frac{\mu}{1 - e^{-\mu}} e^{-\mu \cos\theta} \sin\theta \, d\theta. \qquad (13)$$

The unbiased case corresponds again to $\mu = 0$, but μ can run from $-\infty$ to ∞, where it corresponds to zero string density or a $U(1)/\mathbb{Z}_2$ symmetry, respectively.

Apart from the critical exponents, the interesting feature about biased $\mathbb{R}P^2$ symmetries is that one can bias them to-

TABLE VII. Critical exponents and percolation thresholds for different discretizations of U(1). The values quoted for $\eta^\star$ are taken from the measurement of γ. In the case of the very low Λ, the divergences for loop sizes flatten out long before the critical point and no exponents can be extracted. For very large Λ, however, the number of traced strings is not large enough to extract c. The errors quoted are statistical errors only. NA denotes ''not available.''

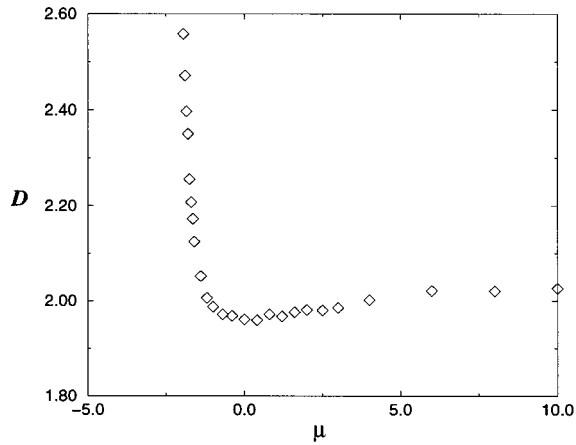| N	Λ	$\eta^\star$	σ	γ	ψ	β
3	50 000	0.279 ± 0.005	0.46 ± 0.02	1.59 ± 0.10	3.79 ± 0.14	0.54 ± 0.10
7	50 000	0.299 ± 0.005	0.44 ± 0.02	1.77 ± 0.14	3.85 ± 0.16	0.45 ± 0.07
15	50 000	0.295 ± 0.007	0.39 ± 0.04	1.86 ± 0.13	4.5 ± 0.3	0.39 ± 0.06
31	50 000	0.305 ± 0.005	0.42 ± 0.02	1.75 ± 0.03	4.1 ± 0.3	0.45 ± 0.06
63	50 000	0.309 ± 0.005	0.417 ± 0.017	1.69 ± 0.07	4.0 ± 0.2	0.50 ± 0.07
127	50 000	0.301 ± 0.004	0.413 ± 0.013	1.80 ± 0.06	4.07 ± 0.14	0.41 ± 0.05
255	2 000		0.339 ± 0.005	NA	NA	NA
255	10^6	0.300 ± 0.002	NA	1.83 ± 0.07	4.10 ± 0.15	0.456 ± 0.025

FIG. 19. Running of the fractal dimension. At zero bias we measure $D = 1.96$, whereas for large positive bias the U(1) measurements are recovered asymptotically. As we approach the zero-density limit (large negative μ), the fractal dimension increases sharply before becoming ill defined. The ensemble has 3000 strings with $\Lambda = 10^5$.

wards the U(1) ensembles, which we have measured already. The fractal dimension then runs from what we have measured for U(1) to values lower than 2. This is shown in Fig. 19.

If this effect is caused by the running of the string density, as comparisons of the relative densities seem to suggest [the average number of penetrated triangles in continuous U(1) is $\frac{1}{4}$, whereas for continuous $\mathbb{R}P^2$ it is $1/\pi$ [25]], our case for arguing that the string statistics quite generally depend on the string density in a way opposed to the tendency polymers show gets further support from this picture. Vachaspati's model itself would be the obvious testing ground for such a hypothesis (the penetration probability per plaquette is $\frac{1}{2}$, the highest known in any model so far), but unfortunately he quotes measurements neither for D nor for b at zero bias and his plots have too large statistical fluctuations to extrapolate towards zero bias. A more detailed analysis of this is given in Fig. 22.

Let us now turn to the measurements of the critical exponents. The average loop size has a very distinct peak at the critical point, immediately allowing a rough estimate of the critical bias at $\mu^\star = -1.875 \pm 0.025$. This is displayed in Fig. 20.

When we measure γ, we get the best fits for $\mu^\star = -1.86 \pm 0.01$ and

$$\gamma = 1.70 \pm 0.06,$$

seemingly in very good agreement with the U(1) data. Also

$$\psi = 3.83 \pm 0.14, \quad \sigma = 0.435 \pm 0.013.$$

To obtain $\langle \rho_\infty \rangle$, we first calculate ρ_{total} with a Monte Carlo integration

$$\rho_{\text{total}} = \int \int \int d\Omega_1 \, d\Omega_2 \, d\Omega_3 \, np(\Omega_1)p(\Omega_2)p(\Omega_3),$$

where $\int d\Omega$ stands for $\int_0^{2\pi} d\phi \int_0^{\pi/2} d\theta$, p is defined by Eq. (13), and n is the string flux as a function of the three pair-
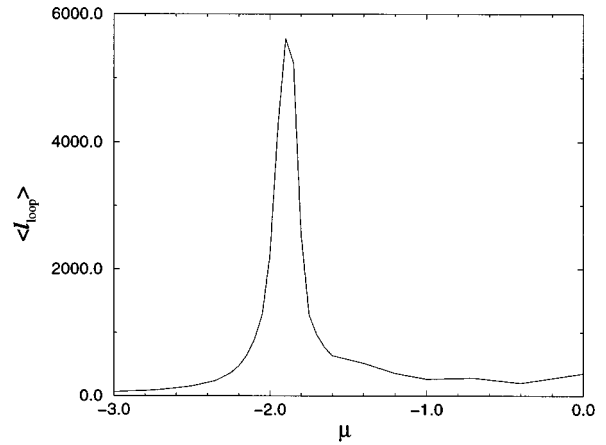

FIG. 20. Loop size distribution for negative μ only. It allows a very good estimate of the position of $\mu^\star$.

wise dot products, as defined in Sec. II C. Then we take again $\rho_\infty = \rho_{\text{total}}(N_\infty / N_{\text{total}})$, where N with a subscript stands simply for the total number of appropriate strings in the ensemble. The results of this integration are plotted in Fig. 21, together with the mass density in loops and infinite strings.

It can be seen that the loop density is considerably lower than for U(1), climbing toward the value measured for U(1) strings as $\mu \to \infty$ and making up the total string density beyond the percolation threshold. There are some statistical errors attached to the lines indicating the loop and infinite string density, as the ensemble of Fig. 21 consists of only 3000 strings ($\Lambda = 100\ 000$).

From the infinite string density we measure

$$\beta = 0.40 \pm 0.03.$$

A comprehensive table of measurements can be found in Table VIII.

Let us now return to a question raised earlier: Is the string density the fundamental parameter, dictating D and b? Figure 22 shows that this cannot be the case. There we plot the density vs. the fractal dimension. Most of the density regime is reached twice, once for negative μ and once for positive
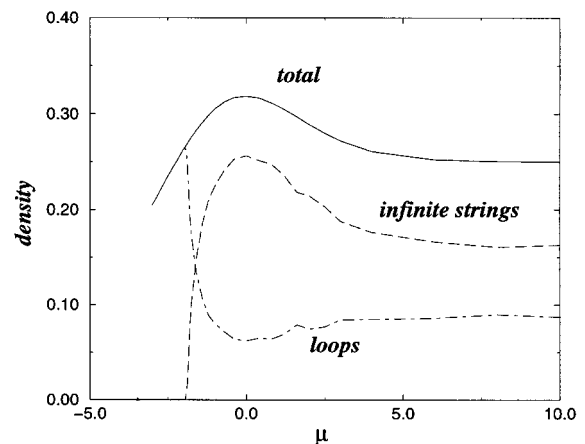

FIG. 21. Density of string segments in the continuous $\mathbb{R}P^2$ ensembles as a function of bias. The density in loops is shown as the dot-dashed line and the density in infinite strings is the dashed line.

TABLE VIII. Summary of all the measured critical exponents, compared with critical exponents in three-dimensional site (or bond) percolation.

Exponent	Percolation theory	Minimally discretized U(1)	Continuous U(1)	Discrete $\mathbb{R}P^2$	Continuous $\mathbb{R}P^2$
σ	0.45	0.46(2)	0.42(2)	0.435(13)	0.40(3)
β	0.41	0.54(10)	0.45(3)	0.41(3)	0.34(4)
γ	1.80	1.59(10)	1.77(6)	1.70(6)	1.68(13)
ψ	4.04	3.79(14)	4.10(15)	3.83(14)	3.75(25)

μ. The fractal dimension, as we go towards the critical point, diverges already for string densities *higher* than the densities we can achieve for positive μ, where the string ensemble just turns into a U(1) set of Monte Carlo data. The fundamental lesson to learn from this is that, although spoiling the symmetry may be the only way to change the string density in our lattice description, it can sometimes be done in different ways and lead to entirely different results.

For the minimal discretization of $\mathbb{R}P^2$ it is straightforward but lengthy to derive

$$\rho_{\text{total}} = \tfrac{1}{18} \left[5 - 3\zeta^2 - 2\zeta^3 \right].$$

It can be seen that it has the right behavior $\rho_{\text{total}} \rightarrow 0$ as $\zeta \rightarrow +1$ and $\rho_{\text{total}} \rightarrow \frac{2}{9}$, which is the value for the minimal discretization of U(1), as $\zeta \rightarrow -1$. It also has an extremum at perfect symmetry $\zeta = 0$. This means that it shows all the qualitative features of the bias dependence we observed in Fig. 21 for the continuous $\mathbb{R}P^2$ strings. The success of discretizations of $\mathbb{R}P^2$ in reproducing qualitatively the behavior of the continuous manifold is quite striking: We have tested another icosahedral discretization, obtained by viewing a vertex of the icosahedron head-on (such that all the other five points are equatorial ones). Again, this reproduces the same qualitative curve for the bias dependence of the density and turns exactly into a representation of the $N = 5$ discretization of U(1) when biased towards equatorial vacuum states (with the same string density). Th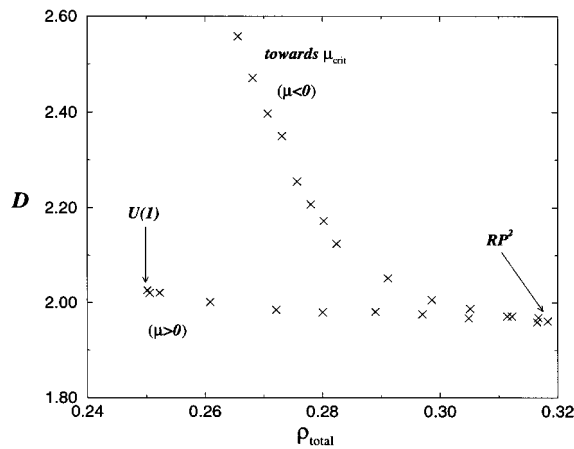e density, together with the density in loops and infinite strings, is shown in Fig. 23. All components have qualitatively the same behavior as in the continuous $\mathbb{R}P^2$ ensembles.

From this we can conclude that, again, the discretization of the vacuum manifold does not spoil the physical picture qualitatively. On sets of 10 000 strings with $\Lambda = 25\,000$, we measure that the percolation threshold is at $\zeta^\star = 0.425 \pm 0.004$, $\sigma = 0.40 \pm 0.03$, $\beta = 0.34 \pm 0.04$, $\gamma = 1.68 \pm 0.13$, and $\psi = 3.75 \pm 0.25$.

What is somewhat unexpected is that for the $\mathbb{R}P^2$ ensemble, violation of the ''strict'' scaling relation Eq. (5) seem to be apparent even in the regime of low bias. Where the bias turns the symmetry into U(1), the strict scaling relation Eq. (5) seems to hold. For the other values, although statistical fluctuations are recognizable, there seems to be a consistent trend towards too large values for b, which disappears again at the critical point [cf. the same discussion for biased U(1) strings]. Perhaps the deviations seen for U(1) in Fig. 18 are also not just statistical errors. An analogous plot for $\mathbb{R}P^2$ strings is shown in Fig. 24 for the discretized manifold and in Fig. 25 for the continuous one.

The obvious source to suspect would be a cutoff dependence for b. However, this is not the case. We have changed upper and lower cutoffs for the loop distribution before fitting it to Eq. (4), and the results do not change qualitatively: b stays recognizably larger than predicted by Eq. (5). Clearly, a better analysis of this violation of scale invariance, even in the percolating regime, is needed. Here, however, we will focus on the nonpercolating regime from now on.
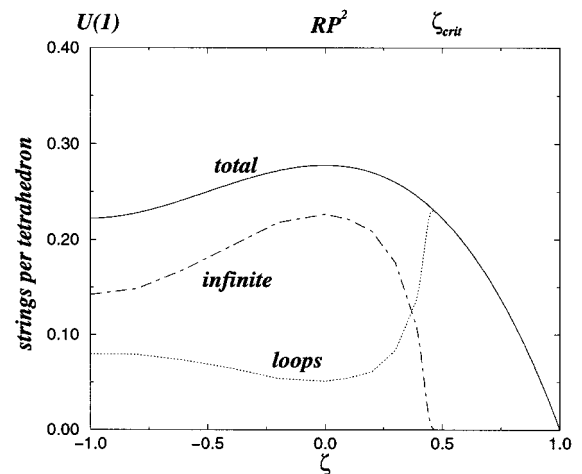

FIG. 22. Density of string segments in the continuous $\mathbb{R}P^2$ ensembles as a function of the total string density. The two ends of the function differ distinctly for positive and negative values of the bias.

FIG. 23. String densities of the discretized $\mathbb{R}P^2$ ensembles. The contributions from loops and infinite strings are shown separately.
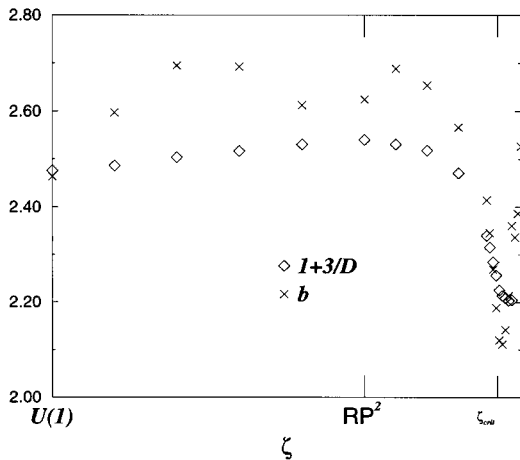
FIG. 24. The scaling relation Eq. (5) seems to be consistently violated, even in the low bias regime, except at the critical point and, perhaps, at a perfect U(1) symmetry. These are the ensembles with the discretized $\mathbb{R}P^2$ manifold. The loop statistics are not extremely good, as the number of loops for zero bias (i.e., where it is the lowest) is ≈ 1800.

V. PERCOLATION THEORY, UNIVERSALITY, AND POLYMERS

A. Percolation theory and critical phenomena

The most notable fact about the critical exponents we have measured in the preceding section is that not only do they agree with each other, thus indicating universal behavior for string defects at the critical density, but they also show reasonable agreement with the exponents obtained by standard site or bond percolation. This fact has already been noticed for the $N=3$ discretization of U(1) strings [28]. We give a comparison of the measured exponents, including their statistical errors, with known results from percolation theory in Table VIII.

It is seen that the continuous symmetries show some reasonable correspondence with percolation data, whereas the minimally discretized U(1) ensemble deviates more strongly. The $\mathbb{R}P^2$ data have critical exponents that are consistently slightly lower than the percolation theory exponents. This
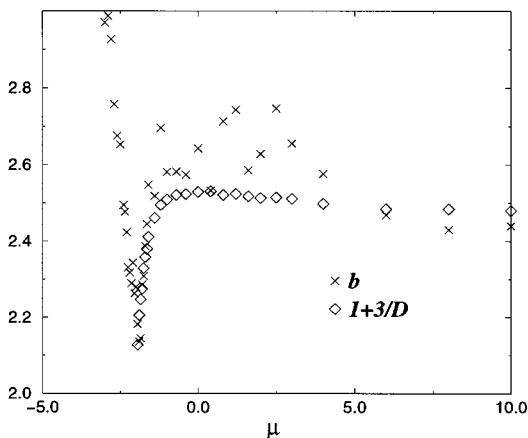

FIG. 25. Same plot as in Fig. 24 for the continuous $\mathbb{R}P^2$ manifold. There are also large statistical errors in b, as the number of loops is extremely small (≈ 500) at low bias.

allows some fairly wide-ranging conclusions. For instance, we can now, without having made the appropriate measurements, expect Vachaspati's model not only to be "nearly" a bond percolation problem as far as the actual percolation threshold is concerned, but it may also have critical exponents very close to the ones of bond percolation.

This is indeed somewhat surprising: From the point of view of percolation theory, we have constructed a very awkward definition of "clusters." Not only are they topologically defined, but they are also definable through continuous, as well as discrete, manifolds of possible lattice states.

There is another comment to be made: From Table VIII, one could hypothesize that there may be a trend for higher-dimensional manifolds to deviate more strongly from percolation data. However, Vachaspati's previously discussed model describes strings produced from a $S^\infty/\mathbb{Z}_2$ ground-state manifold. We have shown above that this model corresponds closely to site percolation, such that one expects especially the Vachaspati model to be close to a percolation problem. Therefore one would not expect the differences between the $\mathbb{R}P^2$ data and standard percolation exponents to be a generic consequence of the higher dimensionality of the manifold.

However, our measurements certainly do not seem to compellingly suggest exact agreement of our critical exponents with bond (or site) percolation in three dimensions. Nevertheless, the relative smallness of the deviations should make no difference when one tries to understand the possible physical relevance of the string statistics near the critical point. Let us therefore try to learn more lessons from percolation theory.

B. Fisher exponent and partition function

The first step towards some indications that there are some universal relations between the configurational exponents (and that, therefore, we may extract more exponents than we have actually measured) has been done already by defining a partition function in Eq. (12), which we can view as a generating function for the loop size distribution, such that

$$\langle l_{\text{loop}}^n \rangle = Z^{-1} \left(-\frac{d}{dc} \right)^n Z.$$

This partition function has the same functional form as the one derived in [34] in the dilute free-string approximation. If we want to return to a physical interpretation of the partition function, it is perhaps natural to expect some of the ingredients of the results of Ref. [34] to enter the description. In this way, one is led to interpret c as an effective string tension σ_{eff}, divided by a temperature, and the Hagedorn transition occurs when the effective string tension (or the effective energy per unit length of string) is zero. Another paper by the same authors [36] then provides the understanding for the mechanism by which the Hagedorn transition occurs in the case of biased symmetries: We have seen from the microscopic details of the lattice description that, for biased symmetries, the strings tend to become more crumpled as they wind around the sparsely spread lattice points with disfavored vacuum values. In [36] it has been shown that for the

reverse situation, when one tries to incorporate stiffness, the effective string tension for fixed temperature behaves as

$$\sigma_{\text{eff}} = \sigma - \frac{\ln[(1+z)e^{-\beta u}]}{l_0 \beta},$$

where u is the energy cost of turning a string through a right angle and z is the number of directions available to the string at every lattice point. From the details of the lattice description (and also from an intuitive point of view in the continuum case), this energy cost becomes negative for strongly biased symmetries, as becomes immediately obvious if one considers the limit in which the disfavored vacuum values occupy only very small regions of space to understand the trend of negative energy cost associated with crumpling the string, so that for some value of the bias $\sigma_{\text{eff}} = 0$ will be achieved and a Hagedorn transition occurs. However, this correspondence is qualitative at best, since we do not deal with string dynamics. For this reason, the exponent b in the partition function, for example, also differs from the one obtained in [34] and the actual string tension right at the phase transition is zero by definition (at least in the case of a second-order phase transition). A more stringent proof that such a thermal interpretation of the partition function yields nonsensical results is developed in Appendix C. Since ''our'' partition function describes the initial conditions for dynamical calculations, one should not view it as being more than the generating function of the moments of the loop length distribution.

If we know how b and c scale at the critical point, we can derive the critical exponents for all the moments of the loop length distribution. From Sec. IV A it is clear that b depends only weakly on the bias, whereas c scales as $c \propto |\eta - \eta^\star|^{-\gamma}$. In agreement with the percolation theory literature, we define $\tau = b(\eta^\star)$ and will call it the Fisher exponent. The critical exponents for the moments of the loop distribution are then derived by

$$\langle l_{\text{loop}}^n \rangle = Z^{-1}\left(-\frac{d}{dc}\right)^n \left[\sum_l l^{-\tau+1} e^{-cl}\right] \propto \sum_l l^{-\tau+1+n} e^{-cl}.$$

Approximating the sum by an integral, we have

$$\langle l_{\text{loop}}^n \rangle \propto \int l^{-\tau+1+n} e^{-cl} dl \propto c^{\tau-2-n} \int z^{2-\tau} e^{-z} dz$$

$$\propto |\eta - \eta^\star|^{(\tau-2-n)/\sigma}, \tag{14}$$

where we have approximated b by the constant τ in the first step and used $c \propto |\eta - \eta^\star|^{1/\sigma}$ in the last step. The integral over z is just a numeric constant. Thus we find that the following scaling relations should hold for our string ensembles:

$$\gamma = \frac{3-\tau}{\sigma}, \quad \psi = \frac{4-\tau}{\sigma}, \tag{15}$$

and so on for higher moments of the loop distribution. In this way, one can use two of the critical exponents as the fundamental ones and derive the others from them. Note that, after the measurements of Sec. IV A, the partition function is no longer just a *model* of how the moments of the loop distri-

TABLE IX. Testing the scaling relations Eqs. (15) and (16) by comparing the values of the Fisher exponent predicted by them.

τ	Minimally discretized U(1)	Continuous U(1)	Discrete $\mathbb{R}P^2$	Continuous $\mathbb{R}P^2$
$\beta\sigma+2$	2.25(6)	2.19(2)	2.18(2)	2.14(3)
$3-\gamma\sigma$	2.27(8)	2.26(6)	2.26(5)	2.33(11)
$4-\psi\sigma$	2.26(15)	2.28(15)	2.33(11)	2.50(22)

bution should scale: It is a strict consequence of the fact that Eq. (8) is satisfied so well by the data. The fact that only *two* critical exponents can be independent is also a consequence of the existence of just *one* crossover length scale. This is discussed in Ref. [23]. The trouble with our definition of a generating function is that it yields the loop length distribution only for $\eta > \eta^\star$. What one is interested in for low bias is the density in infinite strings. The critical exponents thereof cannot be extracted as straightforwardly as the other exponents of the loop distribution, seeing that $\langle l_{\text{loop}}^0 \rangle$ would formally correspond to 1. A prescription that is generally workable in percolation theory is to argue that since

$$\langle l_{\text{loop}} \rangle \propto \frac{d}{dc} \langle l_{\text{loop}}^0 \rangle,$$

we can write

$$\rho_\infty = \text{const}_1 + \text{const}_2 \langle l_{\text{loop}} \rangle c$$

and therefore

$$\rho_\infty \propto |\eta - \eta^\star|^{(\tau-3)/\sigma+1/\sigma},$$

such that we have

$$\beta = \frac{\tau-2}{\sigma}. \tag{16}$$

This can also be made more precise by introducing a general scaling function that is meant to describe both the long- and short-range limits exactly. We chose not to introduce it here since we do not have enough data to extract such a scaling function. One may wonder why this gives the behavior for infinite strings rather than the total mass density or the loop mass density. To answer this, one can just remind oneself that we have extended arguments that hold only for loops at $\eta > \eta^\star$. In this regime, gradually longer and longer loops dominate the loop size distribution such that if we extend the arguments beyond the percolation threshold, it is only the infinite strings that contribute to the moments of this distribution. This can easily be seen by the fact that all moments $\langle l \rangle$ and higher, if defined through the partition function, diverge in the whole percolating regime.

The scaling relations (15) and (16) seem to be very well obeyed by the critical exponents we measure and seem to give a value of τ consistent with what we observe in Fig. 18 and consistent with each other. The different predictions of τ, to make the scaling relations consistent with each other, are listed in Table IX.

C. Average loop size

Encouraged by this, we can go on to another cosmologically perhaps more relevant parameter, which is not directly measurable from our ensembles, since we did not record string-string correlation functions. It is another correlation length, call it χ, that measures the average separation between two points on the same string. This gives a clearer picture of the actual physical size of a collapsed string lump beyond the percolation transition. Define

$$\chi^2 = \frac{\sum\limits_r r^2 g(r)}{\sum\limits_r g(r)},$$

where $g(r)$ is the probability of finding, at a distance r from the origin, a segment of string belonging to the same string as the segment at the origin. If we call the average squared distance between two string segments R_l^2, we have to weigh it with the number of string segments in loops of given length to obtain the same definition for χ in terms of R_l^2,

$$\chi^2 = \frac{2\Sigma R_l^2 n_l l^{2-\tau} e^{-cl}}{\Sigma n_l l^{2-\tau} e^{-cl}}.$$

Thus, apart from numerical prefactors, χ is the radius of those loops that give the main contribution to $\langle l \rangle$. Since we have already established that, at least close to the percolation threshold, Eq. (5) is satisfied and loops have part in a proper definition of the fractal dimension, we can substitute $R_l \propto s^{1/D}$ and the numerator scales, according to Eq. (14), with a critical exponent $(3-\tau+2/D)/\sigma$. The denominator is just $\langle l \rangle$, such that, for the longest loops (i.e., the ones that exhibit fractal behavior on intermediate scales),

$$\chi \propto |\eta - \eta^\star|^{-\nu},$$

with

$$\nu = \frac{1}{D\sigma},$$

where D is now the fractal dimension at criticality $D \approx 2.5$. We have thus a reasonably good understanding of how the statistics of stringy lumps changes as we approach the Hagedorn transition.

One interesting aspect of this may relate, for instance, to axion cosmology: It is known [37,38] that the $U(1)_{PQ}$ Peccei-Quinn symmetry arising in axion models may never have been a perfect symmetry. If this is the case, this may solve the domain wall problem arising in thermal axion scenarios [i.e., scenarios with no inflation below the Peccei-Quinn scale where the Peccei-Quinn symmetry $U(1)_{PQ}$ arises] with color anomaly $N > 1$. The network of axionic domain walls, arising at the QCD scale, bounded by axionic strings, which formed at the Peccei-Quinn scale, may never have been an infinite domain wall network. This would solve the domain wall problem in thermal axion scenarios.

VI. OPEN QUESTIONS

A. Features generic to properties of the vacuum manifold

We have seen that, at perfect symmetry, $\mathbb{R}P^2$ strings and SO(3) strings tend to have lower fractal dimension and therefore higher values of b than the U(1) strings. Within the high statistical errors and considering the extremely low value of Λ used in [30], it is impossible to conclude with certainty that there are significant differences in these parameters between $\mathbb{R}P^2$ strings and SO(3) strings, although the mean value of D measured by Kibble for SO(3) strings is 1.95, i.e., it is lower than for $\mathbb{R}P^2$ strings. It would be worth investigating whether the lower fractal dimension is generic to $\mathbb{Z}_2$ strings, to non-Abelian strings, or indeed, as our hypothesis goes, to the higher string densities one achieves with these symmetry groups. Having data for so few vacuum manifolds only, this question remains unanswered at present.

Furthermore, the fraction of mass in string loops is lower in SO(3) strings ($\approx 5\%$ [30], to be compared with caution, since a cubic lattice was used) than in $\mathbb{R}P^2$ strings ($\approx 19\%$) or U(1) strings ($\approx 35\%$). The same questions as for the symmetry dependence of D and b arise. Nevertheless, the tendency to have fewer loops with increasing string density is apparent. Vachaspati's graphs [25] for the loop density, when extrapolated to zero bias, are extremely close to zero.

One way of addressing this would be an approach similar to the one presented in Fig. 19, starting with a symmetry manifold of high dimensions and transforming it smoothly into several different lower-dimensional symmetries. This is planned for the future. Ideally, one would want some analytic arguments for why such a symmetry-group dependence should occur, if it can be observed consistently for several different vacuum manifolds.

A partial answer can, however, be given right now: Since Vachaspati's model is equivalent to an $\mathbb{R}P^\infty \equiv S^\infty/\mathbb{Z}_2$ symmetry, the sequence $\{U(1) \simeq S^1/\mathbb{Z}_2, \quad \mathbb{R}P^2 \simeq S^2/\mathbb{Z}_2, \quad SO(3) \simeq S^3/\mathbb{Z}_2, \ldots, S^\infty/\mathbb{Z}_2\}$ has been probed to some extent, and it seems that as we move along this series, the string density increases *with* the dimensionality of the vacuum manifold. For continuous representations of the symmetry group, the exact string densities have been calculated by Vachaspati [25].

We now know that, at least among the $\mathbb{R}P^N$ symmetries, the lowest string density and lowest proportion of density in infinite strings is probably achieved for a U(1) symmetry. Now let us remind ourselves that the smallest allowed string loops (in lattice units of the dual lattice) on the tetrakaideka-hedral lattice are probably shorter than on any other sensible regular lattice and the fraction of string mass in loops for a fixed vacuum manifold is therefore likely to be higher on a tetrakeidekahedral lattice than on any other one. We therefore suspect that the measured proportion of infinite string density for U(1) strings is the lowest one can find on any regular lattice and for any $\mathbb{R}P^N$ vacuum manifold. It is certainly the lowest found to date in VV simulations at perfect symmetry on a regular lattice. All vacuum manifolds, other than U(1), with nontrivial π_1 have more dimensions than U(1). If the dimensionality of the vacuum manifold turns out to be the relevant parameter, the measured 63% for U(1) strings would actually be the lowest mass fraction of infinite

strings for *any* vacuum manifold on *any* regular lattice, if measured with the VV algorithm.

However, when comparing these symmetry groups, even at perfect symmetry, one observes that they just continue the trend one observes for spoiled symmetries when moving away from the percolation threshold: As the string density increases, not only the loop density, but also the fractal dimension decreases. This seems to suggest that the total string density may play a fundamental role in deciding the fractal dimension of the strings, even in an unbiased symmetry. We are lacking an analytic argument for this hypothesis, and Fig. 22 introduces the additional complication that biased symmetries do not scale their string statistics the same way as unbiased ones do, although the trend of an increase in the fractal dimension with decreasing density is maintained in either case. Group theory alone does not seem to tell much about where the strings go. Perhaps a renormalization-group description, generalizable to different symmetries, would be useful in shedding some light on this question. This brings us to the next unsolved problem: We do not have *any* renormalization-group description that allows us to put in (and conserve) the string density as the fundamental parameter.

B. Renormalization-group arguments

Having observed that the scaling relation Eq. (5) holds reasonably well through a whole range of values of the bias, we would like to understand how this comes about since percolation theory does not really help us there (except when close to the threshold). Because we are dealing with a critical phenomenon, one would hope that renormalization-group arguments can shed light on this question. In this section we will mention the difficulties with such attempts and why we have been unsuccessful (so far) in finding nontrivial fixed points.

It is straightforward, but tedious, to derive renormalization-group arguments, derived from RG ideas in percolation theory, with vacuum field averaging (at least in the lattice description). The technical parts of this appear in Appendix B. Here we just take the renormalization-group polynomial as given. Let us group the sites of a unit cell of the bcc lattice [i.e., a cell with spanning vectors $(1,0,0)$, $(0,1,0)$, and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$] into a single supersite such that the coarse-grained lattice is again a bcc lattice with twice the lattice spacing of the original lattice $a'=2a$. If we take a U(1) symmetry discretized by $\mathbb{Z}_3$, at arbitrary bias η, such that the probabilities for the three vacuum values are $p(0)=p$ and $p(1)=p(2)=\frac{1}{2}(1-p)$, with a sensible renormalization procedure (see Appendix B), the renormalized bias becomes

$$p'=\frac{105}{8}p^8-55p^7+\frac{315}{4}p^6-\frac{63}{2}p^5-\frac{175}{8}p^4+\frac{35}{2}p^3.$$

The fixed points are the solutions to $p'(p)=p$ and are, as expected, at 0, $\frac{1}{3}$, and 1. Because the renormalization procedure in Appendix B was chosen such that the field values over the group of sites are averaged in some sensible way, this means that the interpretation of the vacuum values, existing at the sites of the tetrahedral lattice, as a horizon-

volume average over the field value is sensible and, even if the description is chosen to be more or less coarse grained than one correlation length per lattice spacing, we would get the same scaling laws for any such lattice description at zero bias ($p=\frac{1}{3}$) or at maximum bias ($p=0,1$, which are trivial fixed points, as no strings appear at all and the field is in its true ground state).

This is comforting in itself, but clearly this renormalization does not preserve the features we have measured for small bias: $p=\frac{1}{3}$ is the *only* nontrivial fixed point, but is unstable ($dp'/dp>0$ at $p=\frac{1}{3}$), which means that *any* series of coarse grainings will end up at a trivial fixed point if we start with any small bias. If this was a sensible discretization procedure, we would have to conclude that large loop sizes are exponentially suppressed for *any* nonzero bias. This renormalization procedure is clearly inappropriate when we try to understand string percolation near the percolation threshold because $p_c\neq\frac{1}{3}$. For $p_c<p<\frac{1}{3}$ the simple averaging will always produce a homogeneous background on large scales and therefore predict no infinite strings at all.

For biases near the string percolation threshold, a meaningful renormalization procedure would have to keep information on the percolation properties of the least likely vacuum value. In that case, however, the obvious approach is to take a standard percolation renormalization. This would mean we gain nothing that we do not know yet, as the best we could do to develop some intuitive understanding of the Hagedorn transition on the lattice was to take percolation theory results, before we even started contemplating about the renormalization group.

However, it turns out that the renormalization group in three-dimensional percolation is never exact [35] and needs systematic improvement, so that the coarse graining does not reconnect disconnected clusters or disconnect connected ones. On some level, this problem will always reappear. Therefore the systematic improvement does not yield an exact renormalization group either. From Fig. 6 it is obvious (at least for Vachaspati's $\mathbb{R}P^\infty$ model) that such systematic improvement would differ, depending on whether we want to improve the renormalization group for a string description or for a bond percolation problem. This explains on a more formal basis why the Vachaspati model is so close to a percolation problem but not quite identical to it.

Clearly, finding a systematically improved renormalization description starting from a percolation picture would enhance our analytic understanding of the Hagedorn transition and associated critical exponents. Depending on how the details of this procedure will turn out to work, it may then also be possible to extend it to groups other than $S^\infty/\mathbb{Z}_2$ and the minimally discretized U(1), for which the percolation theory picture was so effective.

It is not at all obvious what variables should be conserved in such a renormalization-group description. We have seen in this section that preserving vacuum field averages alone is not enough. Towards the end of Appendix B we show that preserving the string flux alone is not enough either.

CONCLUSION

We have investigated how generally known concepts of statistics of topological line defects are affected if a

symmetry-breaking phase transition occurs in such a fashion that the remaining symmetry in the manifold of possible ground states is only approximate. Such a concept has been familiar as a solution to the domain wall problem. There it leads naturally to a percolation theory understanding of defect statistics because the sets of possible ground states are disconnected. So far this concept has not been worked out properly for continuous symmetries, mainly perhaps because the naive correspondence with standard percolation theory breaks down for continuous symmetries.

Out of the variety of possible defects exhibited by non-simply-connected continuous vacuum manifolds, we concerned ourselves in this work with string defects in both perfect and approximate symmetries. With improvements in our algorithm, explained in detail in Ref. [7] and based on the methods of Ref. [6], we have been able to go much further with the measurements presented here and are able to provide answers to a number of important questions.

(i) Is the fractal dimension of infinite strings precisely 2? For infinite strings in the U(1) models the answer, summarized in Table II, is ''yes,'' within the statistical errors of about 0.8% (to reach this accuracy we have used a simple extrapolation to cope with finite-size effects). It should be noted, however, that this is the infinite-length limit of a running effective fractal dimension (compare Ref. [28]). For strings in the $\mathbb{R}P^2$ models (see Table V) the answer appears to be ''no'': we have good evidence from the low-cutoff high-statistics simulations of the continuous $\mathbb{R}P^2$ manifold that the fractal dimension is slightly lower than 2. More evidence is displayed in Fig. 19, which shows that the fractal dimension of discretized $\mathbb{R}P^2$ strings is less than that of U(1) strings.

(ii) Are strings scale invariant in the percolating phase? Scale invariance in our sense means that the loop size distribution follows Eq. (2) and that Eq. (5) is satisfied. Figure 18 shows that the assumption of scale invariance is consistent with our measurements for U(1) strings. However, the evidence for $\mathbb{R}P^2$ strings in Figs. 24 and 25 is against scale invariance, except at the percolation transition. In both cases, the fractal dimension increases as the string density decreases: The string gets more crumpled. This is implicit in Vachaspati's results [25].

(iii) Are the critical exponents of the string percolation transition universal? We have investigated several different representations of two vacuum manifolds U(1) and $\mathbb{R}P^2$, and for these models at least we have found good evidence, summarized in Table VIII, that the answer is ''yes.'' Furthermore, the critical exponents are those of three-dimensional site or bond percolation.

Our claims of universality are aided by the construction of a two-parameter generating function for the loop size distribution, which has some of the qualities of a partition function (but no thermal interpetation). The existence of such a distribution implies that certain scaling relations characteristic of universality should be satisfied, as indeed they are (Table IX).

Our results extend the work of Bradley *et al.* [28] on trichord percolation, who first noted the correspondence of the critical exponents with that of site percolation. With our variety of models of the manifolds, we provide evidence that the correspondence is not confined to their three-color model

[equivalent to our three-point triangulation of U(1)].

The simulations presented here model the initial conditions of condensed-matter systems with a nonconserved order parameter after a rapid quench. Dynamical simulations have been performed on systems where the nonconserved order parameter is a complex scalar field ϕ, both with $\langle\phi\rangle=0$ (corresponding to zero bias) and $\langle\phi\rangle\neq0$ [39]. It is found that the introduction of a bias in the initial expectation value of the order parameter results in the eventual departure from dynamical scaling, with the density of the string network going as

$$\rho(t)\sim t^{-1}\exp(-gt^{3/2}),$$

where g depends approximately quadratically on the initial bias $\langle\phi\rangle$. This is due to the network breaking up into isolated loops, with an exponentially suppressed size distribution. When we published Ref. [6], it was not clear whether this is due to the initial conditions possessing no infinite string or whether the infinite string somehow manages to chop itself up into an infinite number of loops. If it had been the former, the percolation transition would have to happen at very small bias, perhaps even at $\langle\phi\rangle=0$, for the departure from power-law scaling in $\rho(t)$ was observed from rather small biases, down to $\langle\phi\rangle=0.001$. Having extended the work of Ref. [6] to (a) better statistics, (b) much larger upper cutoff lengths, (c) one more symmetry group, and (d) continuous representations of the ground-state manifold, we have now ascertained that this is not the case. The existence of infinite strings holds for a range of biased initial conditions.

By the example of string defects, this work has provided some intuitive, as well as some quantitative understanding of how defects in systems with continuous symmetries are affected by a lifting of the degeneracy of the possible ground states. Since these effects also change the late-time dynamics of systems with nonconserved order parameter, the hope is that understanding the initial conditions carries us one step further in understanding these dynamics. It should also provide some incentive to study other defects under similar conditions, in theory and in the laboratory.

APPENDIX A: FAILURE OF THE DODECAHEDRAL DISCRETIZATION OF $\mathbb{R}P^2$

Here we will prove that a discretization of $\mathbb{R}P^2$ achieved by using only those points that are the vertices of a dodecahedron (DH) embedded into the sphere does not force self-avoidance of the resulting string defects. Let us take the upper half of the dodecahedron, with its vertices numbered according to Fig. 26.

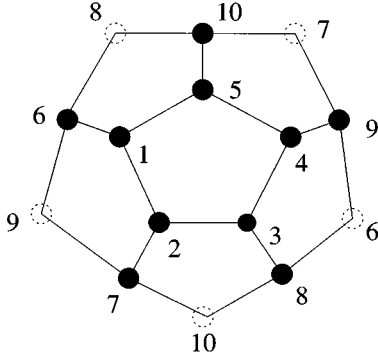Not all the point pairs are directly linked to each other,

FIG. 26. Convention for the numbering of the vertices of the dodecahedron in Appendix A. Points lying on the lower half sphere are drawn as dotted circles. The projections of the points 1–5 to the lower half sphere are not drawn, but all their links can be seen here.

but all are at most two links apart. There are no ambiguities in applying the geodesic rule because no pairs can be connected in more than one way, if we only allow connection lengths of up to two links. For any assignment of such a restricted selection of elements of $\mathbb{R}P^2$ to the vertices of a triangle, we can therefore identify uniquely whether the triangle is penetrated by a string or not. This is equivalent to the statement that we could use the same flux definition as before because none of the projections to the upper half sphere of vectors pointing to the vertices of the DH are at right angles to each other. This correspondence will make the proof easier because we can just check whether a given shortest path between two points crosses the equator and assign it a value of -1 if it does or 1 otherwise. The product of all links will then indicate whether there is a string present in a triangle (if the product is negative) or not. Now let us take, e.g., the assignments 2, 5, 8, and 9 for the respective vacuum fields on the vertices of a tetrahedron. Then the links have the following ''equator-crossing indicators'' assigned to them:

$$(25) \rightarrow +1,$$

$$(28) \rightarrow +1,$$

$$(29) \rightarrow -1,$$

$$(58) \rightarrow -1,$$

$$(59) \rightarrow +1,$$

$$(89) \rightarrow +1.$$

Since the two links that have -1 assigned to them are disconnected, all triangles border exactly one of the two links and all the link-variable products are negative. There are therefore four string segments entering the tetrahedron. ∎

APPENDIX B: RENORMALIZATION GROUPS

1. Vacuum field averaging

Let us group the eight points of the bcc lattice cells spanned by the vectors $(1,0,0)$, $(0,1,0)$, and $(\frac{1}{2},\frac{1}{2},\frac{1}{2})$ such that the most frequently occurring vacuum value will be the value

TABLE X. Configurations of the eight sites grouped together to a supersite, which yield a vacuum value of 0 for the supersite, together with their respective probabilities, according to Eq. (17). It is assumed that $a \neq b$ and neither is equal to 0.

Field values ($+$permutations)	Number of permutations	Probability
(0,0,0,0,0,0,0,0)	1	p^8
(0,0,0,0,0,0,0,a)	8×2	$p^7\left(\frac{1-p}{2}\right)$
(0,0,0,0,0,0,a,a)	27×2	$p^6\left(\frac{1-p}{2}\right)^2$
(0,0,0,0,0,0,1,2)	56	$p^6\left(\frac{1-p}{2}\right)^2$
(0,0,0,0,0,a,a,a)	56×2	$p^5\left(\frac{1-p}{2}\right)^3$
(0,0,0,0,0,a,a,b)	168×2	$p^5\left(\frac{1-p}{2}\right)^3$
(0,0,0,0,1,1,1,1)	70	$p^4\left(\frac{1-p}{2}\right)^4$
(0,0,0,0,a,a,a,b)	280×2	$p^4\left(\frac{1-p}{2}\right)^4$
(0,0,0,0,1,1,2,2)	420	$p^4\left(\frac{1-p}{2}\right)^4$
(0,0,0,1,1,1,2,2)	560	$p^3\left(\frac{1-p}{2}\right)^5$

assigned to the supersite. In case of ambiguities, i.e., if two values occur three times each or four times each, we take the smaller value (for consistency, this means $0<1$, $1<2$, and $2<0$ to preserve the periodicity of the circle). Let us define the respective probabilities for the vacuum field values as

$$p(0)=p, \quad p(1)=p(2)=\frac{1-p}{2}. \tag{B1}$$

The possible configurations yielding, for the supersite, a field value of zero, then have respective probabilities as listed in Table X.

This gives, for the probability of the renormalized supersite to have a zero vacuum field,

$$p' = P(p) = p^8 + 8p^7(1-p) + 28p^6(1-p)^2 + 56p^5(1-p)^3$$
$$+ \frac{525}{8}p^4(1-p)^4 + \frac{35}{2}p^3(1-p)^5$$
$$= \frac{105}{8}p^8 - 55p^7 + \frac{315}{4}p^6 - \frac{63}{2}p^5 - \frac{175}{8}p^4 + \frac{35}{2}p^3. \tag{B2}$$

The polynomial $P(p)-p$ is plotted in Fig. 27. The zeros of this plot correspond to fixed points. Fixed points outside the range [0,1] have no interpretation in terms of probabilities.
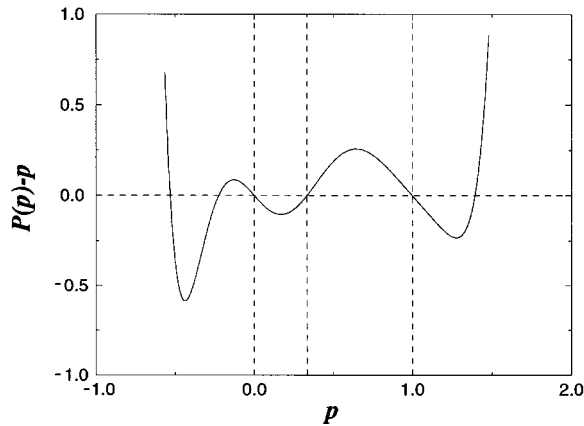
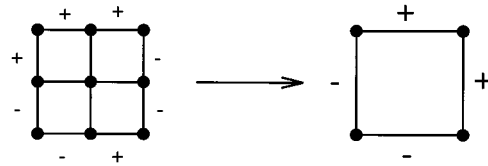FIG. 27. Renormalization polynomial $P(p)-p$ of Eq. (18).

FIG. 28. Renormalization procedure for a grouping such that the coarse-grained lattice spacing is twice the spacing on the original lattice. The assignments to the links inside the finer lattice is irrelevant to the total string flux. Changing any of the outer links of the fine lattice changes the string flux and changes one of the links of the coarse-grained lattice, changing the string flux also in the renormalized description.

$$p' = \sum_{k=0}^{[n/2]} \left[\binom{n}{k} p^{n-k}(1-p)^k \right] - \frac{1}{2} \delta_{2[n/2],n} \binom{n}{n/2} p^{n/2}(1-p)^{n/2}.$$

The bracket $[n/2]$ denotes the largest integer not larger than $n/2$ such that the term containing the Kronecker δ corrects for the double counting of symmetric terms if n is even. Again, the only physical fixed points for all of these polynomials (with $n>2$) are the trivial ones at $p=0,\frac{1}{2},1$ and no percolation threshold can be identified in this way.

2. Renormalizing the Vachaspati model with conserved flux

As the next obvious step, one could try a renormalization procedure that conserves string flux. Clearly, this is most easily done for $\mathbb{Z}_2$ strings, where the string flux can only be 0 or 1 (or better, "even" or "odd"): If the first homotopy group of the vacuum manifold is, e.g., $\mathbb{Z}$, then the string flux through multiply renormalized supersites can become arbitrarily large (if it is to be conserved) such that in every renormalization step the lattice definition of the string flux would have to change together with the probabilities of the relevant parameters. As seen in all the models for $\mathbb{Z}_2$ strings [i.e., the Vachaspati model, our calculations for $\mathbb{R}P^2$, and the calculations of Kibble for SO(3) strings [30]], this is essentially done by identifying lattice links as having one of the two values ± 1 assigned to them and taking the product over a closed contour. The obvious approach is therefore to associate two consecutive links with a superlink and assigning to it the product of the two link values. On any lattice (cubic or tetrahedral), this preserves the string flux in the sense that the string flux through a superplaquette is zero (even) if the sum of the string lines penetrating the constituent plaquettes is even and odd (one) otherwise. The procedure is outlined in Fig. 28.

The third dimension does not have to be drawn since the renormalization procedures for differently oriented links are independent from each other (the second dimension is necessary only to show how changes in the string flux survive the renormalization procedure). Let us define the probability of finding a link with value $+1$ by $p_+ = p$ such that $p_- = 1-p$. Naively, one expects that, as the size of the superplaquettes increases with every renormalization step, the string flux for very large plaquettes is 0 or 1 with equal
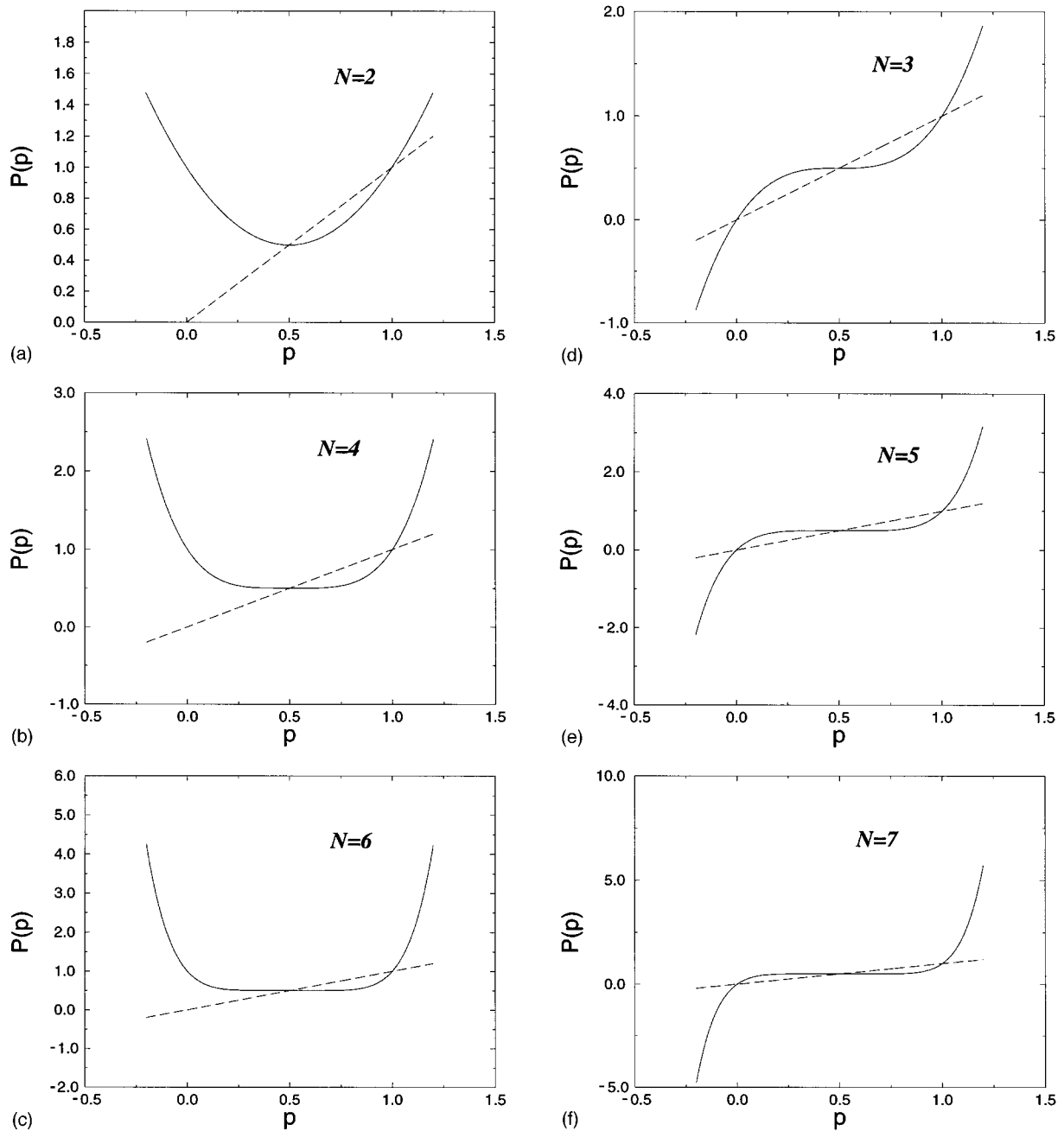
We see that the only physical fixed points are at 0, $\frac{1}{3}$, and 1. They are all expected, as discussed in Sec. VIB. What is not discussed there is that the stability of the fixed points is determined by the sign of $dp'/dp = dP/dp$ at the fixed points. As expected again, the fixed points 0 and 1 are the only stable ones, so that this renormalization procedure would get rid of all infinite strings for any $p \neq \frac{1}{3}$, which is clearly different from our measurements. A renormalization procedure that merely performs field averages is therefore not sufficient to describe the Hagedorn transition in the initial conditions (but it may perhaps describe the disappearance of infinite strings in the subsequent dynamical evolution, as observed in Ref. [39]).

We stress that we have not been able to find any nontrivial fixed points for various generalizations of Eq. (18). It is straightforward to write down the renormalization polynomial for general groupings of n sites to a supersite by generalizing the criteria leading to valid entries in Table X.

$$p' = P^{(n)}(p)$$
$$= \sum_{\substack{l,k=0 \\ n-l \geq 2k \\ n-k \geq 2l}}^{l,k \leq n/2} \frac{n!}{l!k!(n-l-k)!} p^{n-l-k} \left(\frac{1-p}{2} \right)^{k+l}$$
$$\times \left(1 - \frac{1}{2} \delta_{n-k-l,l} - \frac{1}{2} \delta_{n-k-l,k} + \frac{1}{3} \delta_{n,3l} \delta_{l,k} \right),$$

which does not seem to have any physical fixed points other than the trivial ones found with the polynomial in Eq. (18). The Kronecker δ's just divide out factors of 2 or 3, depending on whether the most frequent vacuum phases are more than one, as, e.g., in the fourth to last and last row of Table X. We have tried the same approach for a minimally discretized $\mathbb{R}P^2$ symmetry where it is in fact much easier to write down the renormalization polynomial for the bias, but again all the fixed points are the trivial ones and the percolation threshold cannot be located in this way. For completeness, the easily derivable renormalization polynomial, if $p(0)=p(1)=p(2)=p/3$ and $p(3)=p(4)=p(5)=(1-p)/3$ and if we group a number n of sites to become a supersite of the renormalized lattice, becomes

FIG. 29. Renormalization polynomial for orders 2–7. There is a clear tendency for the potential to flatten out near $p = \frac{1}{2}$, making the speed of convergence towards the stable fixed point $p = \frac{1}{2}$ faster for larger N. There is no tendency for the emergence of any fixed points other than 0 (for odd polynomials only), $\frac{1}{2}$, and 1. The dotted line is the line $p = p$. The intersections define the fixed points.

probability, irrespective of the initial bias, as long as the average number of expected (nonrenormalized) -1 links on the edge of the superplaquette is much greater than 1, i.e., for any nontrivial bias. The probabilities to have -1 or $+1$ assigned to a link converge towards $p_- = p_+ = \frac{1}{2}$, i.e., $p_+ = \frac{1}{2}$, should be a stable fixed point. This is exactly what happens. p' consists of two contributions: both links of the nonrenormalized lattice have to have the same value assigned to them in order to make the superlink carry the value $+1$. Therefore

$$p' = p^2 + (1-p)^2.$$

It is easy to see that the only fixed points are $\frac{1}{2}$ and 1: If the

entire nonrenormalized lattice consists of links with -1 (i.e., $p = 0$), the renormalized lattice will consist entirely of links with $+1$ such that $p = 0$ is not a fixed point. Furthermore, $dp'/dp = 4p - 2$ such that $p = 1$ is an unstable fixed point. $p = \frac{1}{2}$ is a stable fixed point and is approached exponentially fast for any sequence of renormalization steps. This fast approach is also understandable: As soon as the probability of having a value of -1 on any of the original links spanned by a superlink becomes appreciable, the probabilities of having either $+1$ or -1 assigned to the superlink are very nearly equal. The probability of encountering a -1 link approaches 1 exponentially with the size l of the superlink as $1 - p^l$.

This procedure cannot be refined by combining more than two links to a superlink in the next step since the arguments

leading to the stable fixed point $p = \frac{1}{2}$ stay valid for any such renormalization. This has been tested up to coarse grainings by factors of 7. The renormalization polynomial in any of these coarse-graining procedures is (if the lattice is coarse grained by a factor of N)

$$p' = P_N(p) = \sum_{k=0}^{k \leq N/2} \binom{N}{N \bmod 2 + 2k} p^{N-2k}(1-p)^{2k},$$

for which it is trivially true that $p = 1$ is a fixed point. Using

$$1 + \binom{n}{2} + \binom{n}{4} + \cdots = 2^{n-1},$$

$$\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1},$$

it is easy to see that $p = \frac{1}{2}$ is also a fixed point for all of these renormalization procedures. Also, for odd numbers of N, $p = 0$ is a fixed point, as one expects (an odd number of -1 links gets renormalized to a -1 superlink). We have tested $P_N(p)$ for values up to $N = 7$, and the mentioned fixed points are really the only solutions of $P_N(p) = p$ for all of these. Furthermore, the speed of divergence of $P_N(p)$ outside the interval $[0,1]$ tends to increase, so that it seems that the physical fixed points are in fact the only real ones and all other solutions to $P_N(p) = p$ are complex numbers. Surely it should be possible to find a rigorous proof for this statement, from the general form of $P_N(p)$, but instead we refer to Fig. 29, which seems to suggest very strongly a tendency for the $P_N(p)$ not to develop any other physical fixed points as we increase N.

What this means is that the renormalization procedure proposed here is also inappropriate for telling us about the scaling of the string network, as all it tells us is that through very large surfaces (on superhorizon scales), the probability of finding a $\mathbb{Z}_2$-string flux 1 is $\frac{1}{2}$. In particular, it does not enable us to locate a string percolation threshold. Summarizing both sections of this appendix, the question of how to formulate a renormalization-group transformation that identifies the Hagedorn transition point as a fixed point remains unanswered, as does the question of which variables one should preserve in such a transformation.

APPENDIX C: COMPARISON TO THERMAL QUANTITIES

In Ref. [6] we attempted to extract a statistical temperature definition from the probability distribution of strings. This can of course be done only by analogy to statistical-mechanics arguments. It will turn out that the temperature thus obtained is meaningless in physical terms and we arrive at a circular argument, giving us that c should be interpreted as an inverse temperature, which is not the case, as we argued before. Thus comparisons with thermal ensembles or thermal partition functions are not appropriate.

If we define a probability density that a given string segment belongs to a loop in the length interval $[l, l+dl]$ by $P(l) \propto (l \, dn/dl)$, we can define an information entropy, contained in the probability distribution function by

$$S = -\sum_l P(l) \ln P(l).$$

We can, for instance, try to relate the average length of string loops to an average energy of a thermal ensemble. It would then make sense to define a temperature θ as

$$\frac{1}{\theta} = \frac{\partial S}{\partial \langle l \rangle}$$

and see how the system behaves under changes of the bias. In Ref. [6] we did not have good enough statistics to extract that parameter, but with the partition function we do not need to get involved with the numerically highly unstable division of two differentials because, without normalization,

$$P(l) = l^{-\tau+1} e^{-cl}.$$

The entropy can then be split into two parts

$$S = -\sum_l l^{-\tau+1} e^{-cl}[(-\tau+1)\ln l - cl] = S_1 + S_2.$$

Then

$$S_2 \propto c \langle l_{\text{loop}} \rangle \propto c^{\tau-2}.$$

S_1, by partial integration, behaves as

$$S_1 \propto \int \ln(l) l^{-\tau+1} e^{-cl} dl \propto c^{\tau-2}[\text{const}_1 - \text{const}_2 \ln c].$$

Thus, with

$$dS \propto (\Delta \eta)^{(\tau-2-\sigma)/\sigma}[1 + \text{const} \ln \eta] d(\Delta \eta)$$

and

$$d\langle l_{\text{loop}} \rangle \propto (\Delta \eta)^{(\tau-3-\sigma)/\sigma} d(\Delta \eta),$$

we obtain

$$\theta \propto (\Delta \eta)^{-1/\sigma}[1 + \text{const} \ln \eta]$$

and the thus obtained temperature diverges, up to logarithmic corrections, as $1/c$ and the Hagedorn transition happens at infinite θ. Thus the correspondence of our partition function with a thermal one does not hold. ∎

[1] P.G. de Gennes, *Scaling Concepts in Polymer Physics* (Cornell University Press, Ithaca, 1979).

[2] T.W.B. Kibble, J. Phys. A **9**, 1387 (1976).

[3] A. Vilenkin and E.P.S. Shellard, *Cosmic Strings and other Topological Defects* (Cambridge University Press, Cambridge, 1994).

[4] M.B. Hindmarsh and T.W.B. Kibble, Rep. Prog. Phys. **58**, 477 (1995).

[5] P. Ferreira and N. Turok (unpublished).

[6] M. Hindmarsh and K. Strobl, Nucl. Phys. B **437**, 471 (1995).

[7] K. Strobl, Ph.D. thesis, Cambridge University, 1995 (unpublished) (http://starsky.pcss.maps.susx.ac.uk/users/karls/thesis/index.html).

[8] J. Borrill, Phys. Rev. Lett. **76**, 3255 (1996).

[9] J. Robinson and A. Yates, Phys. Rev. D **54**, 5211 (1996).

[10] K. Strobl, Sussex University Report No. SUSX-TH-96-014, 1996 (unpublished).

[11] Y.B. Zel'dovich, I.Y. Kobzarev, and L.B. Okun, Phys. Lett. **50B**, 340 (1974).

[12] Ya.B. Zel'dovich, I.Yu. Kobzarev, and L.B. Okun, Zh. Eksp. Teor. Fiz. **67**, 3 (1974) [Sov. Phys. JETP **40**, 1 (1975)].

[13] D. Coulson, Z. Lalak, and B. Ovrut, Phys. Rev. D **53**, 4237 (1996).

[14] P.H. Frampton and T.W. Kephart, Phys. Rev. D **51**, 1 (1995).

[15] R. Leese and T. Prokopec, Phys. Lett. B **260**, 27 (1991).

[16] K. Strobl, Sussex University Report No. SUSX-TH-96-012, 1996 (unpublished).

[17] T. Vachaspati and A. Vilenkin, Phys. Rev. D **30**, 2036 (1984).

[18] S. Rudaz and A.M. Srivastava, Mod. Phys. Lett. A **8**, 1443 (1993).

[19] M. Hindmarsh, A.-C. Davis, and R. Brandenberger, Phys. Rev. D **49**, 1944 (1994).

[20] E. Copeland, D. Haws, T.W.B. Kibble, D. Mitchell, and N. Turok, Nucl. Phys. B **298**, 445 (1988).

[21] R.A. Leese and T. Prokopec, Phys. Rev. D **44**, 3749 (1991).

[22] B. Li, N. Madras, and A.D. Sokal, New York University Report No. NYU-TH-94-09-01, 1994 (unpublished).

[23] K. Strobl (unpublished).

[24] A.D. Sokal, in *Monte Carlo and Molecular Dynamics Simulations in Polymer Science*, edited by Kurt Binder (Oxford University Press, Oxford, 1994).

[25] T. Vachaspati, Phys. Rev. D **44**, 3723 (1991).

[26] R.J. Scherrer and J.A. Frieman, Phys. Rev. D **33**, 3556 (1986).

[27] A. Vilenkin, in *The Very Early Universe*, edited by G. Gibbons, S.W. Hawking, and S.T.C. Siklos (Cambridge University Press, Cambridge, 1982).

[28] R.M. Bradley, P.N. Strensky, and J.-M. Debierre, Phys. Rev. A **45**, 8513 (1992).

[29] I. Chuang, R. Dürrer, N. Turok, and B. Yurke, Science **251**, 1336 (1991); I. Chuang, B. Yurke, A.N. Pargelis, and N. Turok, Phys. Rev. E **47**, 3343 (1993).

[30] T.W.B. Kibble, Phys. Lett. **166B**, 311 (1986).

[31] R. Hagedorn, Nuovo Cimento Suppl. **3**, 147 (1965).

[32] M.J. Bowick and L.C.R. Wijewardhana, Phys. Rev. Lett. **54**, 2485 (1985).

[33] D. Mitchell and N. Turok, Nucl. Phys. B **294**, 1138 (1987).

[34] E. Copeland, D. Haws, and R. Rivers, Nucl. Phys. B **319**, 687 (1989); E. Copeland, D. Haws, S. Holbraad, and R. Rivers, Physica A **179**, 507 (1991).

[35] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1991).

[36] E. Copeland, D. Haws, S. Holbraad, and R. Rivers, Phys. Lett. B **236**, 49 (1990).

[37] H. Georgi, S. Glashow, and M. Wise, Phys. Rev. Lett. **47**, 402 (1981).

[38] M. Kamionkowski and J. March-Russell, Phys. Lett. B **282**, 137 (1992); R. Holman *et al.*, *ibid.* **282**, 132 (1992).

[39] M. Mondello and N. Goldenfeld, Phys. Rev. A **45**, 657 (1992).